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# ELECTRIC CIRCUIT THEORY

**(Part two)**

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This textbook is the continiuation of the university course “Electric Circuit Theory (Part one)” issued by “Publishing House, Technical University” in Tbilisi in 2000. It meant for students of power engineering, communication and information technologies faculties of technical universities.

It includes theory of non-sinusoidal current circuits, transient analysis, electric circuits with impulse drivers, transmission lines.

It covers also theory of two port networks and electric filters.

Examples and problems are given in each section.

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**CONTENTS**

## 1. Theory of Electric Circuits of Non-sinusoidal Periodical Currents

### 1.1. Introduction

This chapter deals with electric circuits where currents and voltages are periodic but of a waveform other then sinusoidal.

There are several reasons those causes non-sinusoidal currents in electric circuits.

Electric circuits made up of linear elements carry a non-sinusoidal current on being connected to a source of non-sinusoidal voltage or emf.

The waveform of the emf produced by an alternator is liable to be a true sinusoidal, but because of the flux wave set up in the air gap of the alternator slightly differs from the sinusoidal due to tooth ripples. The last one imposes on the flux wave by the stator and rotor core slots and for other technical reasons.

Linear electric circuits carry also a non-sinusoidal current on being connected to a source of sinusoidal voltage or emf in the case when a circuit parameter R, L or C varies in time.

Non-sinusoidal currents and voltages appear in a circuit containing non-linear circuit elements even if the circuit is supplied from a source of sinusoidal voltage. For example the current in an iron-cored coil becomes non-sinusoidal since the magnetic flux produced by the coil varies non-linearly with the coil current.

### 1.2. Representation of a Non-sinusoidal Function by the Fourier Series

The waveform of a non-sinusoidal current, voltage or emf in general is shown in

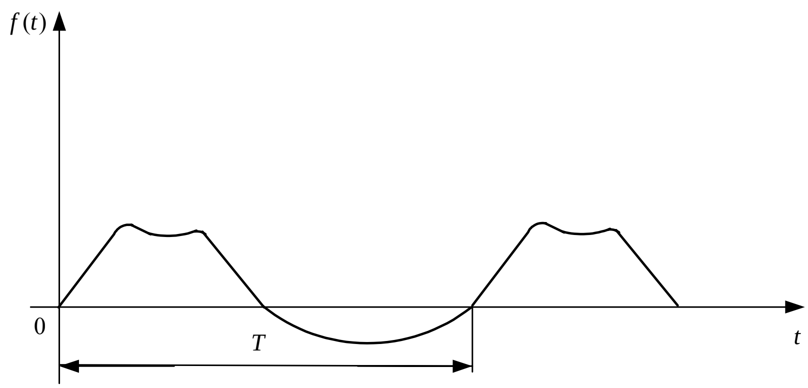



Fig. 1.1

The function f (t) is periodical. It means that this function satisfies the condition

$$f(t+T) = f(t).$$

As it’s known from mathematics, the Fouries series theorem allows a non-sinusoidal function f (t) to represent by the sum of a constant term and sinusoidal functions of multiple frequencies called the harmonic components, or simply harmonics.

$$f(t) = A_0 + A_1 \sin(\omega t + \psi_1) + A_2 \sin(2\omega t + \psi_2) + \cdots A_k \sin(k\omega t + \psi_k) + \cdots , \qquad 1.1$$

where $A_0$ is the constant term of the series,

$A_1$, $A_2$, $A_k$ is the amplitudes of the harmonics,

$\psi_1$, $\psi_2$, $\psi_k$ are the initial phase angles of the harmonics.

The first harmonic has a period equal to that of the non-sinusoidal function and is referred to as the first or fundamental harmonic.

The other harmonics whose frequencies are multiples of the fundamental frequency are called higher harmonics.

The Fouries series may be written in another form. Taking into account that

$$A_k \sin(k\omega t + \psi_k) = A_k \sin k\omega t \cos\psi_k + A_k \cos k\omega t \sin\psi_k = B_k \sin k\omega t + C_k \cos k\omega t, \cdots \qquad 1.2$$

where

$$B_k = A_k \cos \psi_k$$
$$C_k = A_k \sin \psi_k \qquad 1.3$$

the non-sinusoidal function may be expressed as

$$f(t) = A_0 + B_1 \sin \omega t + B_2 \sin 2\omega t + B_3 \sin 3\omega t + \cdots + B_k \sin k\omega t + \cdots +$$
$$+ C_1 \cos \omega t + C_2 \cos 2\omega t + C_3 \cos 3\omega t + \cdots + C_k \cos k\omega t + \cdots \qquad 1.4$$

**1.3. Calculation of Fourier Series Factors**

Factors $A_0$, $B_k$ and $C_k$ of series can be found from the following equations:

$$A_0 = \frac{1}{T}\int_0^T f(t)dt \qquad 1.5$$

$$B_k = \frac{2}{T}\int_0^T f(t)\sin k\omega t dt \qquad 1.6$$

$$C_k = \frac{2}{T}\int_0^T f(t)\cos k\omega t dt \qquad 1.7$$

Using $\alpha = \omega t$ as an argument instead of "t" the formulas (1.5,1.6,1.7) are transformed as

$$A_0 = \frac{1}{2\pi}\int_o^{2\pi} f(\alpha)d\alpha \qquad 1.8$$

$$B_k = \frac{1}{\pi}\int_0^{2\pi} f(\alpha)\sin \alpha d\alpha \qquad 1.9$$

$$C_k = \frac{1}{\pi}\int f(\alpha)\cos \alpha d\alpha \qquad 1.10$$

If it's desired to resolve the non-sinusoidal function into a series of the 1.1 form $A_k$ and $\Psi_k$ may be found using formulas 1.3

$$A_k = \sqrt{B_k + C_k}, \qquad 1.11$$

$$tn\,\psi_k = \frac{C_k}{B_k}. \qquad 1.12$$

When finding angle $\Psi_k$ special attention should be paid to the sign before $C_k$ and $B_k$ as it determines the value of the angle. For example, a positive sign before $C_k$ and $B_k$ gives a positive ratio, and the angle will be in the first quarter. With $C_k$ and $B_k$ negative their ratio will also positive, but in this case the angle lies in the third quarter.

As may be seen from equations 1.5 or 1.8 the constant term of the series represents the average value of the function taken over its period.

Example 1.1. Express the series

$$v(t) = 20 + 5\sin \omega t + 5\sin 2\omega t + 8\sin 3\omega t + \ldots.. - 5\cos \omega t + 8\cos 3\omega t + \ldots$$

in terms of sine terms only.

Solution: We have $V_0 = 20$; $B_1 = 5$; $B_2 = 5$; $B_3 = 8$; $C_1 = -5$; $C_3 = 8$.

Therefore

$$A_1 = \sqrt{B_1^2 + C_1^2} = \sqrt{5^2 + 5^2} = 7.07,$$
$$A_2 = B_2 = 5,$$
$$A_3 = \sqrt{B_3^2 + C_3^2} = \sqrt{8^2 + 8^2} = 11.3.$$

Then

$$tn\psi_1 = \frac{C_1}{B_1} = \frac{-5}{5}, \qquad \psi_1 = -45^0;$$

$$tn\psi_2 = \frac{C}{B_2} = \frac{0}{5}, \qquad \psi_2 = 0;$$

$$tn\psi_3 = \frac{C_3}{B_3} = \frac{8}{8}, \qquad \psi_3 = 45^0.$$

Therefore the series is

$$v(t) = 20 + 7.07\sin(\omega t - 45^0) + 5\sin(2\omega t) + 11.3\sin(3\omega t + 45^0) + \ldots$$

### 1.4. The Grapho-analitic Method of Calculation of the Factors of a Fourier Series

Factors $A_0$, $B_k$, $C_k$ of the Fourier series represented by equations 1.5, 1.6, 1.7 or 1.8, 1.9, 1.10 can be found to a good approximation by the grapho-analytic method.

This is achieved by representing a function to be resolved a table form. For this a period of the curve must be divided into "n" number of the equal intervals as shown in Eig.2.2

Then the values of the function are defined by erecting the ordinates of the curve at equal intervals. The factor $A_0$ can be found from the following approximate equation

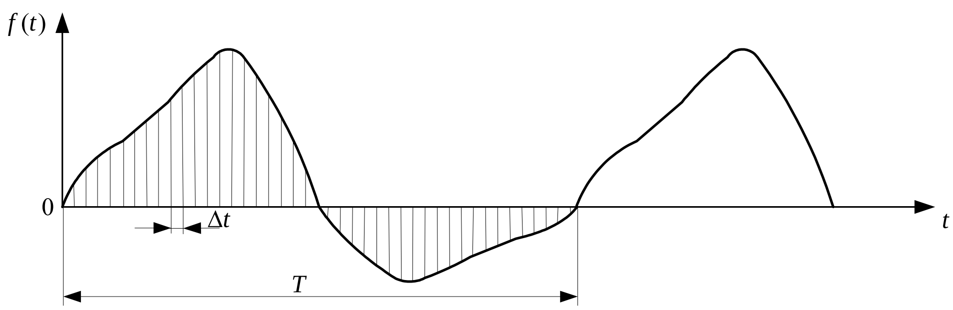



Fig. 1.2

$$A_0 = \frac{1}{T}\int_0^T f(t)dt \approx \frac{1}{n}\sum_{p=1}^{n} f(p\frac{T}{n}), \qquad 1.13$$

where

$$\sum_{p=1}^{n} f(p\frac{T}{n})$$

is the algebraic sum of the ordinates taken over one period of the function.

Similarly, factors $B_k$ and Ck can be found from equations 1.6 and 1.7

$$B_k = \frac{2}{T}\int_0^T f(t)\sin k\,\omega t dt \approx \frac{2}{n}\sum_{p=1}^{n} f(p\frac{T}{n})\sin(kp\frac{\omega T}{n}) \qquad 1.14$$

$$C_k = \frac{2}{T}\int_0^T f(t)\cos k\omega t dt \approx \frac{2}{n}\sum_{p=1}^{n} f(p\frac{T}{n})\cos(kp\frac{\omega T}{n}). \qquad 1.15$$

The result will be better the smaller the interval Δt.

## 1.5. Some Properties of Symmetric, Periodic Waveforms

There are some waveforms in Fig.1.3a,b,c which have certain specific properties.

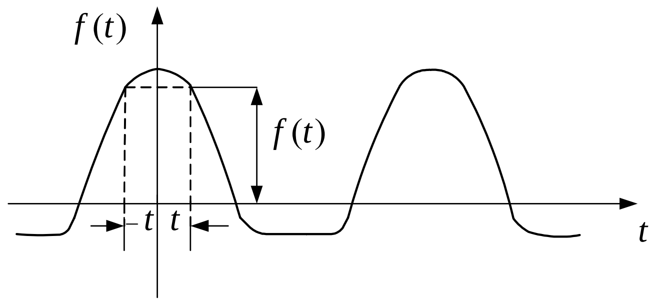



Fig. 1.3a

Thus, the waveform of Fig. 1.3a is described by the equation

$$f(t) = f(-t) . \qquad 1.16$$

Such functions are called as the even functions. This waveform is symmetrical about the ordinate axis. If the curve to the left of the ordinate axis is transposed about this axis to produce its mirror image, the latter will coincide with the curve lying to the right of the ordinate axis. The wave is said to have even symmetry.

The Fourier series of such function contains only the cosine terms and the constant component and no sine terms

$$f(t) = A_o + C_1 \cos\omega t + C_2 \cos 2\omega t + \cdots + C_k \cos k\omega t + \cdots, \qquad 1.17$$

where

$$A_0 = \frac{2}{T}\int_0^{T/2} f(t)dt \qquad 1.18$$

$$C_k = \frac{4}{T}\int f(t)\cos k\omega t dt \ . \qquad 1.19$$

The waveform of Fig. 1.3b is described by the equation

$$f(-t) = - f(t) \qquad . \quad 1.20$$

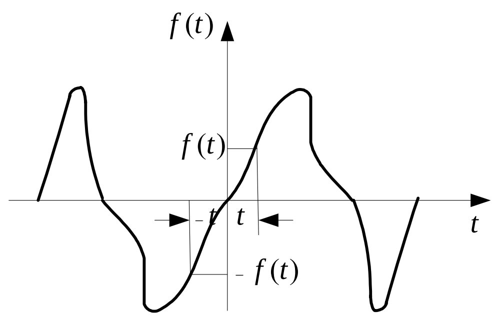



Fig. 1.3b

Such functions are called as the odd functions. This waveform is symmetrical about the origin of coordinate. The wave is said to have odd symmetry. Its Fourier series has no constant term and no cosine terms

$$f(t) = B_1 \sin\omega t + B_2 \sin 2\omega t + B_3 \sin 3\alpha\omega t + \cdots + B_k \sin k\omega t + ..., \qquad 1.21$$

where

$$B_k = \frac{4}{T}\int_0^{T/2} f(t)\sin k\omega t dt \ . \qquad 1.22$$

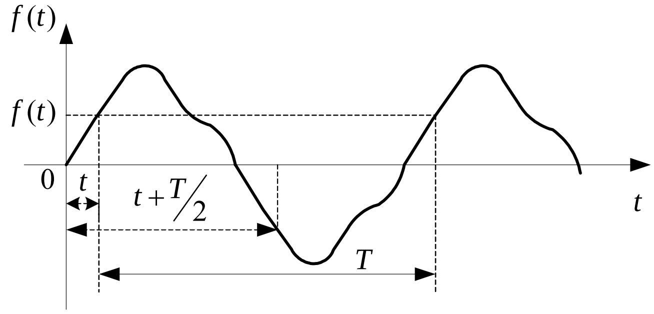



Fig. 1.3c

The waveform of Fig. 1.3c is described by the equation

$$f(t+\frac{T}{2}) = -f(t) \qquad 1.23$$

and is symmetrical about the time axis.

A specific property of this wave is that it changes its sign every half-cycle and if we move the negative waveform by half a period along the time axis towards the origin, we get a mirror image of the positive waveform. The wave is said to have half-wave symmetry.

The Fourier series of such function contains no constant term and no even harmonics.

$$f(t) = B_1 \sin\omega t + B_3 \sin 3\omega t + B_5 \sin 5\omega t + \cdots + B_{2k+1}\sin(2k+1)\omega t + \cdots +$$
$$+ C_1 \cos\omega t + C_3 \cos 3\omega t + C_5 \cos 5\beta\omega t + \cdots + C_{2k+1}\cos(2k+1)\omega t + \cdots \ , \qquad 1.24$$

where

$$B_{2k+1} = \frac{4}{T}\int_0^{T/2} f(t)\sin(2k+1)\omega t dt, \qquad 1.25$$

$$C_{2k+1} = \frac{4}{T}\int f(t)\cos(2k+1)\omega t dt. \qquad 1.26$$

The Fourier series of the trapezoidal waveform of Fig. 1.4 contains only sine terms odd harmonics.

$$f(t) = \frac{2A_m T}{\pi^2\tau}(\sin\omega\tau\sin\omega t + \frac{1}{3^2}\sin 3\omega\tau\sin 3\omega t + \frac{1}{5^2}\sin 5\omega\tau\sin 5\omega t + \cdots), \qquad 1.27$$

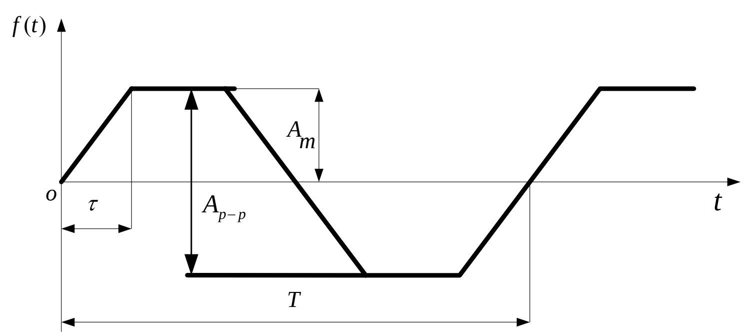



Fig. 1.4

where

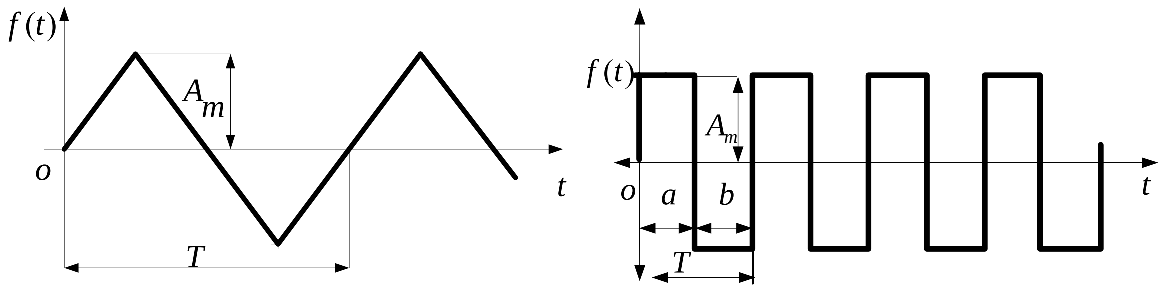
$A_m$ - is the peak value of the function,

$A_{p-p}$ – is the peak to peak value,

T – is the period of the function,

τ - is pulse rising time.

The Fourier series of the triangular waveform of Fig. 1.5 a




a.) b.)

Fig. 1.5

may be derived from the equation 1.27 when τ = T/4

$$f(t) = \frac{8A_m}{\pi^2}(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \cdots) \qquad 1.28$$

The Fourier series of the rectangular waveform of Fig. 1.5 b may be derived from the equation 1.27 if put τ = 0

$$f(t) = \frac{2A_m\omega T}{\pi^2}(\frac{\sin\omega\tau}{\omega\tau}\sin\omega t + \frac{1}{3}\frac{\sin 3\omega\tau}{3\omega\tau}\sin 3\omega t + \frac{1}{5}\frac{\sin 5\omega\tau}{5\omega\tau}\sin 5\omega t + \cdots) =$$

$$= \frac{4A_m}{\pi}(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \cdots) \qquad 1.29$$

because of

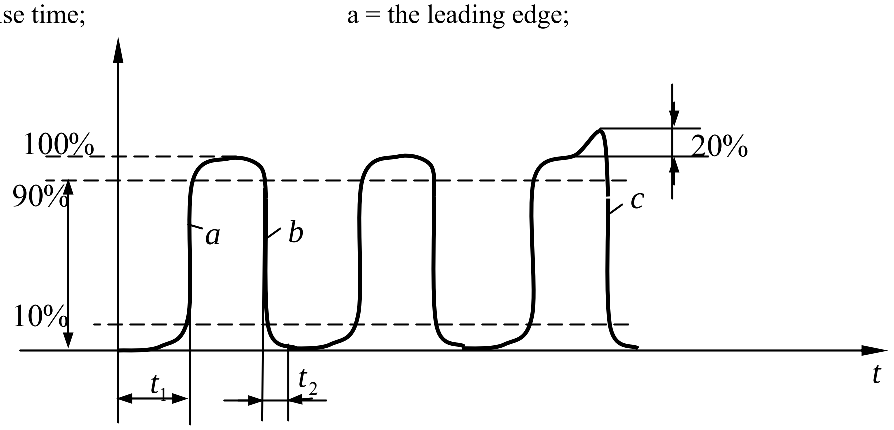
$$\lim\frac{\sin k\tau}{k\tau} = 1$$

The main characteristics of the waveform of Fig.1.5b are also the mark (a), the space (b) and the mark to space ratio (n).

The Fig.1.6 shows some waveforms and their characteristics often used in electrical engineering. They are the real rectangular (a), saw tooth (b), ringing (c) waveforms, ripple voltage (d) and damped oscillations (e).

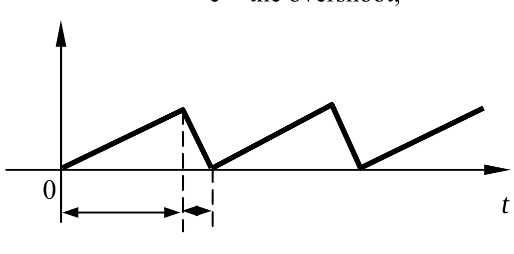
a.)

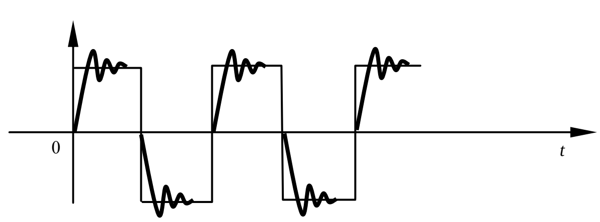
$t_1$= the rise time; a = the leading edge;




$t_2$ = the fall time; b = the trailing edge;

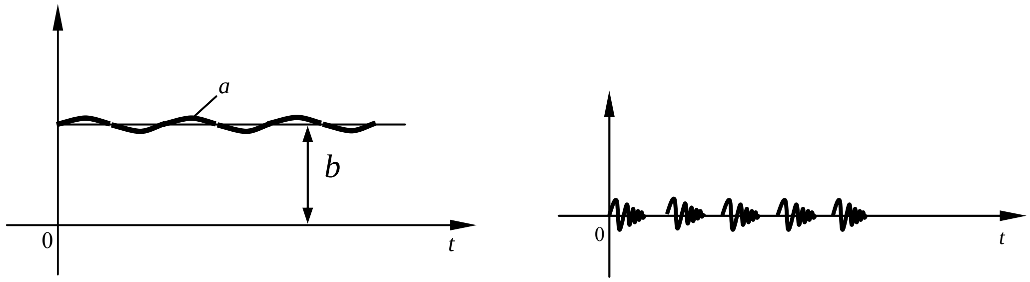
c = the overshoot;




b.) c.)

$t_1$= the sweep time; $t_2$= the fly back




d.) e.)

a = the oscillations
b = the DC component

Fig.1.6

### 1.6. Beats and Modulated Waves

Beats are periodic variations that result from the superposition of two or more waves having equal amplitudes and different but very close frequencies.

Consider the superposition of two sinusoidal waves slightly differing in frequencies $\omega_1$ and $\omega_2$ and having the same amplitude A

$$f(t) = A\sin\omega_1 t + A\sin\omega_2 t. \qquad 1.30$$

From mathematics

$$f(t) = A\sin\omega_1 t + A\sin\omega_2 t. \qquad 1.31$$

and f( t ) may be re-written thus

$$f(t) = 2A\cos\Omega t\sin\omega t. \qquad 1.32$$

where

$$\Omega = \frac{\omega_2 - \omega_1}{2} \quad and \quad \omega = \frac{\omega_2 + \omega_1}{2}, \qquad (\Omega << \omega). \qquad 1.33$$

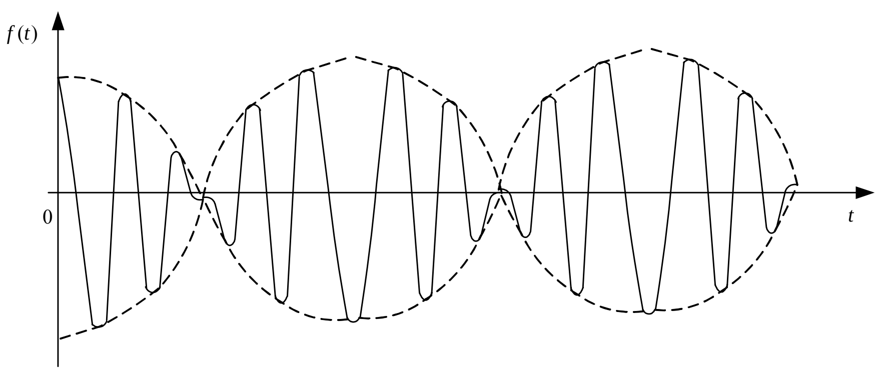



Fig.1.7

The result of the superposition is shown in Fig.1.7. The amplitude of the resultant wave is $2A\cos\Omega t$. The dotted curve enclosing the peaks is the envelope.

In practice, beating is employed in various application. One of them is the determination that two given waveforms differ in frequency.

Modulated waves are widely employed in communication. A modulated wave is a combination of two or more waves, which results in the production of frequencies not present in the original waves. Generally, a modulated wave is described by the equation

$$f(t) = A\sin(\omega t + \psi), \qquad 1.34$$

where the amplitude A, the frequency $\omega$ and the phase angle $\Psi$ are varied separately or together in time.

If only the amplitude A of wave is varied, the new wave is said to be amplitude-modulated, and the method is referred to as amplitude modulation.

If only the phase $\Psi$ of the carrier wave is varied, the new wave is referred to as phase-modulated, and the method is called phase modulation.

If only the angular frequency ω of the carrier is varied, the new wave is referred to as frequency-modulated, and the process is known as frequency modulation.

In the simplest case the amplitude of the carrier is modulated so that it obeys the sine law

$$f(t) = A_0(1 + m\sin\Omega t)\sin(\omega t + \psi), \qquad 1.35$$

where $\Omega \ll \omega$ is the modulating frequency and he modulation factor defined as the ratio of the maximum departure of the envelope from carrier level to the carrier amplitude. As a rule m<1. The envelope is shown by the dotted line in Fig. 1.8

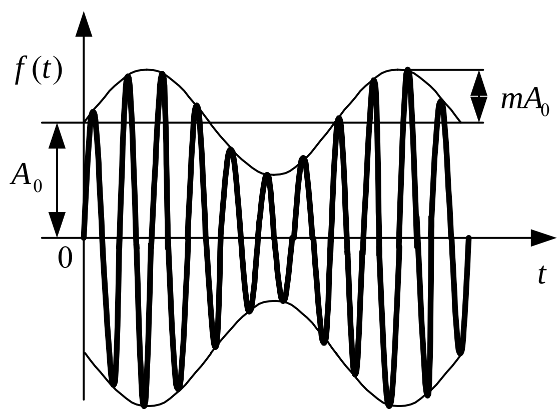



Fig. 1.8

From mathematics it's known that

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta)). \qquad 1.36$$

Therefore, the wave

$$A_0(1 + m\sin\Omega t)\sin(\omega t + \psi)$$

may be expressed as the sum of three waves

$$f(t) = A_0\sin(\omega t + \psi) + \frac{mA_0}{2}\cos[(\omega - \Omega)t - \psi] - \frac{mA_0}{2}\cos[(\omega + \Omega)t + \psi]. \qquad 1.37$$

The values ω -Ω and ω +Ω produced by the process of modulation and lying on both sides of the carrier frequency ω are termed the upper sideband and the lower sideband, respectively.

### 1.7. Frequency Spectrum of a Non-sinusoidal Wave

The frequency spectrum of a non-sinusoidal periodic wave is a plot of harmonic amplitude versus frequency. The plot is in the form of a bar graph or line graph as shown in Fig.1.9a and Fig.1.9b.

The wave of Fig.8.9a has half-wave symmetry and corresponding Fourier series contains only odd harmonics (sine terms). It's rather smooth curve and has a rapidly converging series. That is, only the first few harmonics are dominated; the amplitude of the higher-order harmonics decreases rapidly with increasing frequency. On the other hand, periodic waves with sharp bands such as the saw tooth wave shown in Fig.8.9b have a slowly converging series; such waves have many dominant harmonics.

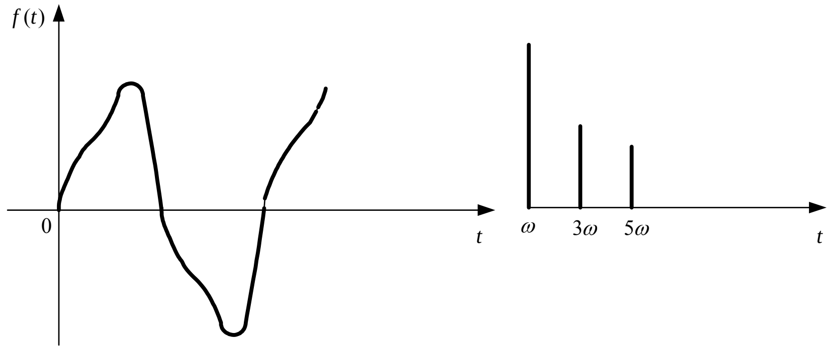



Fig.1.9a

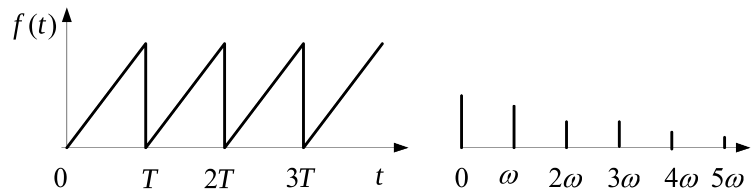



Fig.1.9b

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### 1.8. RMS and Average Values of Non-sinusoidal Current, Voltage and EMF

By definition the rms value of an alternating current

$$I = \sqrt{\frac{1}{T}\int_0^T i^2 dt}. \qquad 1.38$$

If the instantaneous current is expressed by the Fourier series

$$i = I_0 + I_{1m}\sin(\omega t + \psi_1) + I_{2m}\sin(2\omega t + \psi_2) + \cdots + I_{km}\sin(k\omega t + \psi_k)\cdots \qquad 1.39$$

then

$$i^2 = I_0^2 + \sum_{k=1}^{\infty} I_{km}^2 \sin^2(k\omega t + \psi_k) + \sum_{p=0, q=0, p\neq q}^{\infty} I_{pm} I_{qm} \sin(p\omega t + \psi_p)\sin(q\omega t + \psi_q). \qquad 1.40$$

But

$$\int_0^T \sin^2(k\omega t + \psi_k)dt = \int_0^T \frac{1}{2}\left[1 - \cos 2(k\omega t + \psi_k)\right]dt = \frac{1}{2}T - \frac{1}{2}\int_0^T \cos 2(k\omega t + \psi_k)dt = \frac{T}{2} \qquad 1.41$$

and

$$\int_0^T \sin(p\omega t + \psi_p)\sin(q\omega t + \psi_q)dt = 0. \qquad 1.42$$

Therefore

$$I = \sqrt{I_0^2 + I_1^2 + I_2^2 + \cdots + I_k^2 + \cdots} = \sqrt{\sum_{k=0}^{\infty} I_k^2}. \qquad 1.43$$

Thus, the rms value of a non-sinusoidal current is the square root of the sum of the squared values of the direct component and the rms values of the individual harmonic components. The rms value is independent of the phase angles $\psi_k$.

Similarly the rms values of voltage and emf are defined

$$V = \sqrt{\sum_{k=0}^{\infty} U_k^2}. \qquad 1.44$$

$$E = \sqrt{\sum_{k=0}^{\infty} E_k^2}. \qquad 1.45$$

The average value of a non-sinusoidal current is given by equation

$$I_{av} = \frac{1}{T}\int_0^T |i(t)| dt. \qquad 1.46$$

If current waveform has a half-wave symmetry and it does not change the sign during a half period the average value of non-sinusoidal current may be found by equation

$$I_{av} = \frac{2}{T}\int_0^{T/2} i(t)dt. \qquad 1.47$$

If a non-sinusoidal current does not change the sign during a period the average value of current is equal to the direct component

$$I_{av} = I_0. \qquad 1.48$$

In order to analyze an electric circuits of non-sinusoidal current several factors are used. For example the form factor

$$K_f^v = \frac{V}{V_{av}}, \qquad 1.49$$

the factor of distortion

$$K_d^v = \frac{V_1}{V}, \qquad 1.50$$

the peak factor

$$K_p^i = \frac{I_{max}}{I} \qquad 1.51$$

and the harmonic factor

$$K_h^i = \frac{\sqrt{\sum_{k=2}^{\infty} I_k^2}}{I_1}, \qquad 1.52$$

where $V_1$ , $I_1$ are the rms values of the first harmonics of voltage and current, $V_{max}$, $I_{max}$ are the maximum values of voltage and current. The above factors may be written connection with non-sinusoidal emf also.

### 1.9 Power for Circuits of Non-sinusoidal Current

The average power of an electric circuit of a non-sinusoidal current is

$$P = \frac{1}{T}\int_0^T pdt = \frac{1}{T}\int_0^T vidt.. \qquad 1.53$$

Expanding "v" and "i" into Furrier series, we get

$$v = V_0 + V_{1m}\sin(\omega t + \psi_{v1}) + V_{2m}\sin(2\omega t + \psi_{v2}) + ................$$
$$i = I_0 + I_{1m}\sin(\omega t + \psi_{i1}) + I_{2m}\sin(2\omega t + \psi_{i2}) + ..................$$

So the active power is

$$P = \frac{1}{T}\int_0^T \left[V_0 + V_{1m}\sin(\omega t + \psi_{v1}) + V_{2m}\sin(2\omega t + \psi_{v2}) + ..\right]\times$$
$$\times\left[I_0 + I_{1m}\sin(\omega t + \psi_{i1}) + I_{2m}\sin(2\omega t + \psi_{i2}) + ..\right]$$

Because of

$$\int_0^T \sin(p\omega t + \psi_{vp})\sin(q\omega t + \psi_{iq})dt = 0$$

$$\int_0^T \sin(k\omega t + \psi_{vk})\sin(k\omega t + \psi_{ik})dt = \frac{1}{2}\int_0^T \cos(\psi_{vk} - \psi_{ik})dt -$$
$$-\int_0^T \cos(2k\omega t + \psi_{vk} + \psi_{ik})dt = \frac{T}{2}\cos\varphi_k,$$

where

$$\varphi_k = \psi_{vk} - \psi_{ik}$$

is the phase displacement of the k-th harmonic of voltage and current. The formula 1.53 may be written as

$$P = V_0 I_0 + V_1 I_1 \cos\varphi_1 + V_2 I_2 \cos\varphi_2 + ......... + V_k I_k \cos\varphi_k + ... \qquad 1.54$$

Thus the average (active) power of a circuit of non-sinusoidal currents and voltages is equal to the sum of the average powers of the individual harmonics and dc component.

In the case when there is no constant term in the Fourier series of voltage and current it's possible to use the conceptions of the reactive and apparent powers

$$Q = V_1 I_1 \sin\varphi_1 + V_2 I_2 \sin\varphi_2 + ....... + V_k I_k \sin\varphi_k + .....$$

$$S = VI = \sqrt{(V_1^2 + V_2^2 + ..... + V_k^2 + ..)(I_1^2 + I_2^2 + .. + I_k^k + ...)} \quad . \qquad 1.55$$

It must be noted that the apparent power

$$S > \sqrt{P^2 + Q^2}$$

and the distortion power "T` " is usually introduced

$$S = \sqrt{P^2 + Q^2 + (T`)^2}.$$

In dealing with some of the most elementary properties of nonlinear network it's customary to substitute sinusoidal waveforms for non-sinusoidal ones. This is done so that the rms values of the sinusoidal current and voltage would be equal to the rms values of the non-sinusoidal current and voltage being replaced. The phase angle $\varphi_{eq}$ between the equivalent sinusoidal voltage and current is determined from the ratio

$$\cos\varphi_{eq} = \frac{P}{VI}, \qquad 1.56$$

which is the power factor; so that the active power 'P' be the same in both cases.
Example:

$$v = 25.9\sin(\omega t - 11^0 40^`) + 6\sin(3\omega t + 53^0 50^`) \quad (V)$$

$$i = 3\sin(\omega t - 40^0) + 0.9\sqrt{2}\,\sin(3\omega t + 125^0) \quad (A)$$

Solution:

$$V = \sqrt{\left(\frac{25.9}{\sqrt{2}}\right)^2 + \left(\frac{6}{\sqrt{2}}\right)^2} = \sqrt{18.3^2 + 4.26^2} = 18.55 \ (V)$$

$$I = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + 0.9^2} = \sqrt{2.13^2 + 0.9^2} = 2.31 \quad (A)$$

$$\varphi_1 = -11^0 40^` - (-40^0) = 28^0 20^` \quad \varphi_3 = 53^0 50^` - 125^0 = -71^0 10^`$$

$$P = 18.3 * 2.13\cos 28^0 20^` + 4.26 * 0.9\cos(-71^0 10^`) = 35.5 \ (W)$$

$$Q = 18.3 * 2.13\sin 28^0 20^` + 4.26 * 0.9\sin(-71^0 10^`) = 14.84 \ (Var)$$

$$S = VI = 18.55 * 2.31 = 42.8 \ (VA)$$

$$\cos\varphi_{eq} = \frac{35.5}{18.55 * 2.31} = 0.828.$$

### 1.9 Calculation of Electric Circuits of Non-sinusoidal Voltages and Currents

Before getting down to the actual calculation, the emf or applied voltage should be represented as a Fourier series (Fig1.10)

According to the superposition principle, the instantaneous current of any branch in a network equals to the sum of instantaneous currents due to the various harmonics. Similarly, the instantaneous voltage across a section of a network equals to the sum of instantaneous voltages due to the various harmonics across that part. For each harmonic the current and voltage are found by the method discussed in previous chapters.

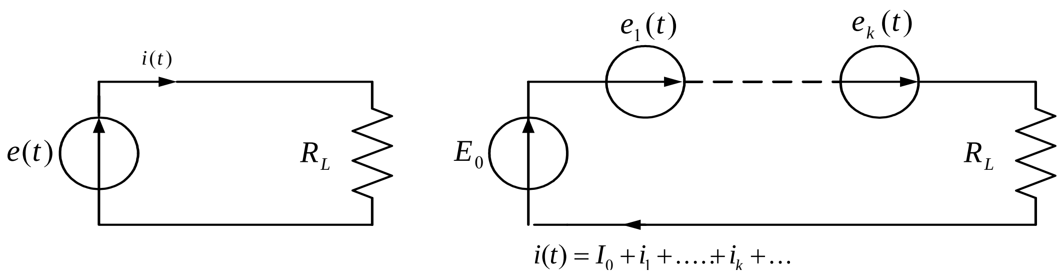



Fig.1.10

As the first step, one finds the currents and voltages due to the constant component of the emf. Then they are found for the first harmonic, then for the second harmonic and so on.

Let's illustrate this method by the following example of Fig.1.11. The applied voltage v (t) is non-sinusoidal periodical value. It's necessary to fined current i (t).

First of all it's necessary to resolve the applied voltage in Fourier series:

$$v(t) = V_0 + V_{1m}\sin(\omega t + \psi_{v1}) + V_{2m}\sin(2\omega t + \psi_{v2}) + ...V_{km}\sin(k\omega t + \psi_{vk}) + ...$$

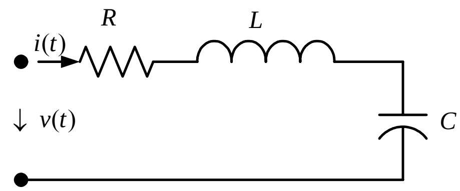



Fig.1.11

Then it is supposed that the voltage $V_0$ is applied to the input terminals of the circuit. Due to the resistance of a capacitor in the case of constant voltage is infinite value DC component of the current

$$I_0 = \frac{V_0}{\infty} = 0$$

At the second step it is supposed that the first harmonic component of voltage is applied. The corresponding component of the circuit current is

$$i_1 = I_{1m}\sin(\omega t + \psi_{v1} - \varphi_1),$$

where

$$I_{1m} = \frac{V_{1m}}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}}, \qquad tn\varphi_1 = \frac{\omega L - \frac{1}{\omega C}}{R}.$$

Then it is supposed that the second harmonic component acts and corresponding component of current is

$$i_2 = I_{2m}\sin(\omega t + \psi_{v2} - \varphi_2),$$

where

$$I_{2m} = \frac{V_{2m}}{\sqrt{R^2 + (2\omega L - \frac{1}{\omega C})^2}}, \qquad tn\varphi_2 = \frac{2\omega L - \frac{1}{2\omega C}}{R}.$$

and so on. The k-th harmonic component of current is

$$i_k = I_{km}\sin(k\omega t + \psi_{vk} - \varphi_k),$$

where

$$I_{km} = \frac{V_{km}}{\sqrt{R^2 + (k\omega L - \frac{1}{k\omega C})^2}}, \qquad tn\varphi_k = \frac{k\omega L - \frac{1}{k\omega C}}{R}$$

The circuit current will be

$$i(t) = I_0 + I_{1m}\sin(\omega t + \psi_{v1} - \varphi_1) + I_{2m}\sin(2\omega t + \psi_{v2} - \varphi_2) + ... + I_{km}\sin(k\omega t + \psi_{vk} - \varphi_k) + ....$$

In the case of more sophisticated circuit (Fig.1.12) it's better to use the complex method.

Suppose the applied voltage is

$$v(t) = V_0 + V_{1m}\sin(\omega t + \psi_{v1}) + V_{2m}\sin(2\omega t + \psi_{v2}) + ... + V_{km}\sin(k\omega t + \psi_{vk}) + ...$$

In the case of the $V_0$ component of voltage $X_{L1}=X_{L3}=0$, $X_{C2} = \infty$ and

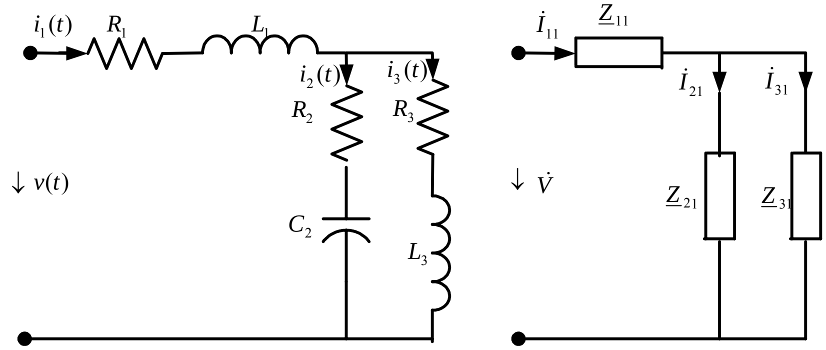



Fig.1.12

$$i_{10} = i_{30} \frac{V_0}{R_1 + R_3}; \; i_{20} = 0.$$

For calculating the harmonic components of currents we have to introduce complex voltages, complex currents and complex impedances. For the first harmonic the circuit diagram is shown in Fig.8.12b. The complex applied voltage is

$$\dot{V}_1 = \frac{V_{1m}}{\sqrt{2}} e^{j\psi_{v1}}.$$

The complex impedances are

$$\underline{Z}_{11} = R_1 + j\omega L_1; \quad \underline{Z}_{21} = R_2 + \frac{1}{j\omega C_2}; \quad \underline{Z}_{31} = R_3 + j\omega L_3.$$

The equivalent complex impedance for the first harmonic is

$$\underline{Z}_{eq,1} = \underline{Z}_{11} + \frac{\underline{Z}_{21}\underline{Z}_{31}}{\underline{Z}_{21} + \underline{Z}_{31}}$$

The complex current of the first section of circuit is

$$\dot{I}_{11} = \frac{\dot{V}_1}{\underline{Z}_{eq,1}} = I_{11} e^{j\varphi_{11}}.$$

In the case of k-th harmonic component

$$\dot{V}_k = \frac{V_{km}}{\sqrt{2}} e^{j\psi_{vk}}.$$

The complex impedances are

$$\underline{Z}_{1k} = R_1 + jk\omega L_1; \quad \underline{Z}_{2k} = R_2 + \frac{1}{jk\omega C_2}; \quad \underline{Z}_{3k} = R_3 + jk\omega L_3.$$

The equivalent complex impedance is

$$\underline{Z}_{eqk} = \underline{Z}_{1k} + \frac{\underline{Z}_{2k}\underline{Z}_{3r}}{\underline{Z}_{2k} + \underline{Z}_{3k}}.$$

The complex current is

$$\dot{I}_{1k} = \frac{\dot{V}_k}{\underline{Z}_{egk}} = I_{1k} e^{j\varphi_{1k}}.$$

The searching current is

$$i_{11} = i_{10} + I_{11m}\sin(\omega t + \varphi_{11}) + I_{12m}\sin(2\omega t + \varphi_{12}) + .... + I_{1km}\sin(k\omega t + \varphi_{1k}) + ...$$

It must be noted that because of the complex harmonic components of current have different frequencies it is not possible to take their sum. In order to find the searching current it is necessary to take the sum of instantaneous values of harmonic components.

### 1.10. Electrical Filters

It is not difficult to note that the reactances of an inductor and a capacitor depend on the frequency. That's why the reactive elements and the way of their connection influence the waveform of voltage and current. Because of the resistance of a resistor does not depend on the frequency the current flowing through the resistor has the same wave form as the waveform of the applied voltage. But in the case of an inductor the reactance at k-th harmonic k time more then the reactance at basic harmonic component and the amplitude of k-th harmonic of current is k-th time less then the amplitude of k-th harmonic of voltage. That's why that the waveform of current slightly differs from sinusoid (Fig.1.13b), when the waveform of voltage is rather strong non-sinusoid.

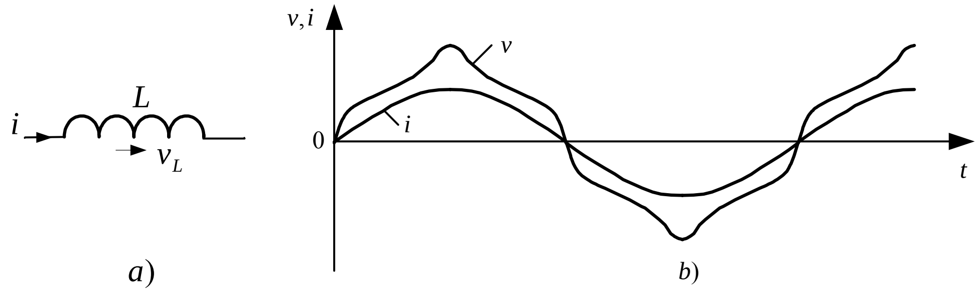



Fig. 1.13

In the case of a capacitor the effect is reverse.

Such behavior of reactive elements is used to construct special kind of electric circuits so called electrical filters. Filter circuits are two-port networks used to block or pass a specific range of frequencies.

If L and C in the circuit of Fig.1.14a are chosen so that $q\omega L=1/q\omega C$, then this section has the zero reactance for q-th harmonic component and the q-th harmonic component of current flows through the load. But in the case of the circuit of Fig.1.14b if $1/q\omega L=q\omega C$ the suseptance of the section is equal to zero and corresponding reactance is infinite. So the circuit blocks the q-th harmonic through the load.

The filter of Fig.1.14c blocks all harmonics and passes only DC component through the load $R_L$. $v_1$ is a pulse voltage that to be obtained by the sinusoidal voltage being rectified. The filters parameters $L_1$, $C_1$, $L_2$, $C_2$, . . .$L_k$, $C_k$ are chosen so that provide the conditions of voltage resonance for harmonic components

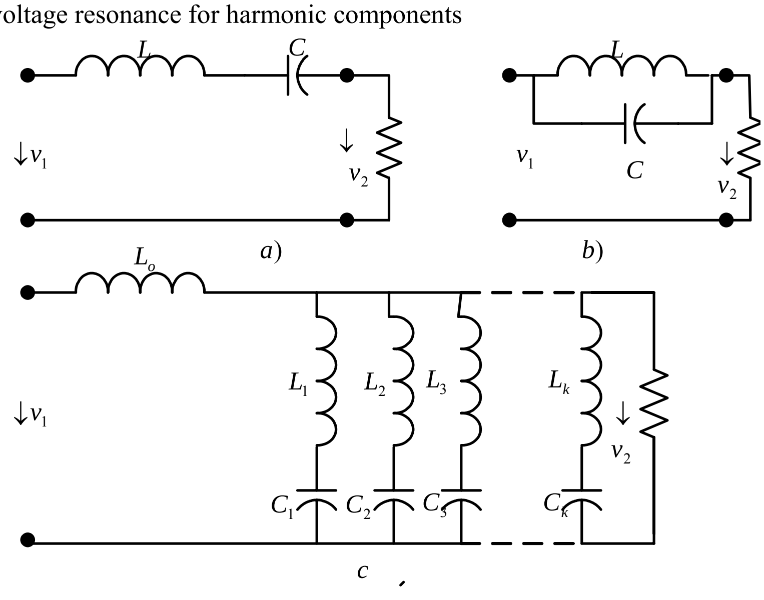



Fig.1.14

$$\omega L_1 = \frac{1}{\omega C_1}; \quad 2\omega L_2 = \frac{1}{2\omega C_2}; \ \ldots\ldots\ldots. k\omega L_k = \frac{1}{k\omega C_k}.$$

For those harmonic components corresponding branches have zero reactance and that's why they never pass through the load.

**Problems**:

1.1. Find the trigonometric form of Fourier series of half-sinusoidal wave form.

1.2. Find the Fourier series for the periodic wave of full wave rectifier.

1.3. A voltage

$$v(t) = 250\sin\omega t + 50\sin(3\omega t + 60^0) + 20\sin(5\omega t + 150^0)$$

is applied to a circuit of resistance 20Ω and inductance 0.05H in series. Derive an expression for the current, the rms value of the current, the active power and the power factor, if the angular frequency ω=314rad/s.

1.4. A nonsinusoidal voltage represented by the expression

$$v(t) = 200\sin\omega t + 20\sin 3\omega t$$

is applied to a coil of 25Ω resistance and 0.02H inductance connected in series with a capacitor of 40μF capacitance. Derive an expression for the current and calculate the power factor of the circuit if the frequency f=50Hz.

## 2. Transient Performance of Linear Electric Circuit

### 2.1. Introduction

The circuit analysis presented in the preceding chapter dealt only with the steady-state behavior of electric circuits that were driven by a DC, sinusoidal or non-sinusoidal periodical sources. This chapter deal with the complete behavior of the circuit for all time which occur between two permanent or steady-state conditions. These steady-state conditions as a rule are periodical processes differing from each other in, as, peak value, phase, wave- form or frequency of EMF, the parameters or configuration of the circuit.

Transient phenomena are brought about in electric circuit by switching operations, that is, by closing or opening switches. There may be however transients due to other causes, such as faults.

In diagram the switching operation is shown by an arrow. Thus the arrow in Fig.2.1a shows that the switch is to be closed and the one of Fig.2.1b is to be opened.

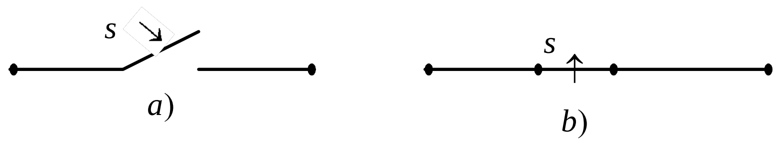



Fig.2.1

Transients usually last a few tenths, hundredths or even millionths of second. It is seldom that they may last several seconds. Yet, their study is very important, as it shows in advance what dangerous rises in voltage or current (many times more their steady-state values) may happen in the sections of a circuit. Transient analysis also shows how signals are distorted in wave-form or amplitude as they pass through amplifiers, filters or other circuit elements.

Transient processes or, simply, transients are electromagnetic processes taking place in electric circuits at transition from one steady-state condition to another.

The time in the course of which the transient takes place within the circuit is known as transient period.

The currents and voltages varying during the transient period are referred to as transient currents and transient voltages.

### 2.2. The Rules of Switching

There are two rules of switching in electric circuit theory. The first rule applies to Inductive circuits. It states that the current flowing through an inductance cannot change instantaneously. In consequence, the instantaneous value of current in an inductive circuit will be exactly the same at the initial instant of the transient period as at the final instant of the preceding steady-state condition

$$i_L(-0)=i_L(+0),$$

where $i_L(-0)$ is the current through the inductance just prior to switching and $i_L(+0)$ is the current through the same inductance immediately after switching. Time $t=-0$ is the instant immediately before switching and $t=+0$ is the time just after switching (Fig.2.2.).

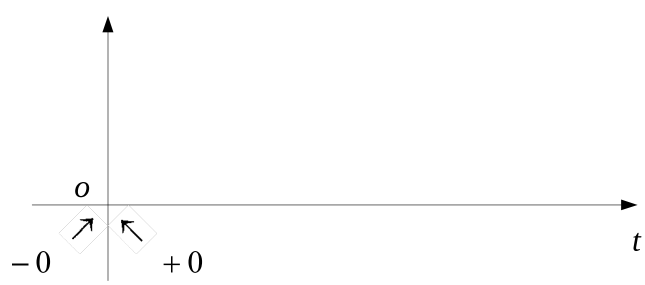



Fig.2.2

The second rule of switching applies to capacitive circuits. It states that the voltage across the capacitance cannot change instantaneously. In consequence, the instantaneous value of the voltage across the capacitance will be the same at the initial of the transient period as at the final instant of the preceding steady-state condition

$v_c(-0)=v_c(+0)$.

Suppose the current flowing through the inductance get the increment $\Delta i$ at switching. Then the energy stored in the magnetic field is increased by the value $\Delta Wm=L\Delta i^2/2$. It takes place at time $\Delta t=0$. So the power $P_L=\Delta W/0=\infty$ is infinite value. But there is no common sense in the infinite power and there is no possibility to increase (or decrease) the current flowing through the inductance instantaneously.

Similarly in the case of capacitance the energy stored in the electric field cannot be changed instantaneously $\Delta W_E=C\Delta v^2/2$ at switching operation because the infinite power is necessary for this $P_C=\Delta W/0=\infty$. So the voltage across the capacitance cannot be changed instantaneously.

### 2.3. General Outline of Transient Analysis (the Classical Method)

Transient analysis as applied to any linear circuit involves basically the following steps:

1. Draw the circuit diagram that to be obtain after switching operation and assume the positive directions for the branch currents.
2. Write network equations using Kirchhoff`s laws.
3. Transform the set of network equations to the one equation, where the fundamental variable is a branch current of the k-th branch. In general form the equation is:

$$a_n \frac{d^n i_k}{dt^n} + a_{n-1} \frac{d^{n-1} i_k}{dt^{n-1}} + \ldots + a_1 \frac{di_k}{dt} + a_0 i_k = f(t), \qquad 2.1$$

where $a_n, a_{n-1}, \ldots a_0$ - are coefficients, f(t) – is the time function that depends on the parameters of energy sources and the circuit elements.

4. Solve the differential equation.

The equation (2.1) is the linear non-homogeneous differential of n-th order. The solution of this equation is the response or transient current of k-th branch

$$i_k(t) = i_k^{`} + i_k^{``}, \qquad 2.2$$

where $i_k^{`}(t)$ - is the forced or steady-state current after switching. This value is the particular solution of the equation (2.1). $I_k^{``}$ - is the general solution of the equation (2.1). It is called as a free component of transient current. It exists only during the transient period.

The free component of the transient current is the solution of the following homogeneous differential equations

$$a_n \frac{d^n i_k^{``}}{dt^n} + a_{n-1} \frac{d^{n-1} i_k^{``}}{dt^{n-1}} + \ldots + a_1 \frac{di_k^{``}}{dt} + a_0 i_k^{``} = 0. \qquad 2.3$$

The solution of the equation (2.3) - the free component

$$i_k^{``}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \ldots\ldots\ldots + A_n e^{\alpha_n t}, \qquad 2.4$$

where $A_1, A_2, \ldots\ldots A_n$ - are the integration constants,
$\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ - are the roots of the following characteristic equation

$$a_n \alpha^n + a_{n-1}\alpha^{n-1} + \ldots\ldots\ldots\ldots a_1 \alpha + a_0 = 0 \qquad 2.5$$

It is essential to be able to determine the order of the characteristic equation by inspection of the network being analyzed for transient performance. This ability helps to estimate in advance the amount of calculations involved and also to reveal any error in the characteristic equation. The order of the characteristic equation is equal to the number of major independent initial conditions after the network has been switched and simplified, and independent of the form of the excitation.

The simplification means that series and parallel inductances are replaced by a single equivalent inductance, and series and parallel capacitances are likewise substituted for by a single equivalent capacitance.

Referring to circuit of Fig.2.3 the series inductances $L_1^{`}$ and $L_1^{``}$ should be replaced by $L_1= L_1^{`} + L_1^{``}$, the series resistances $R_1^{`}$ and $R_2^{``}$ b $R_1=R_1^{`}+R_2^{``}$ and the capacitances $C_3^{`}$, $C_3^{``}$, $C_4$ by

$$C_3 = C_4 + \frac{C_3^{`} C_3^{``}}{C_3^{`} + C_3^{``}}.$$

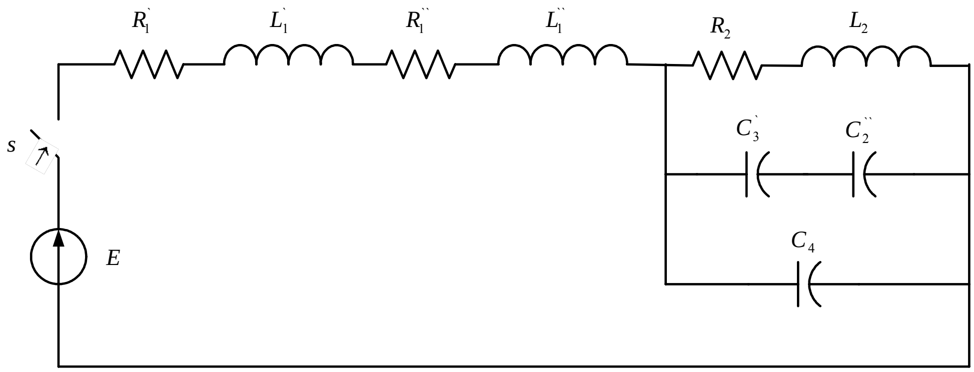



Fig.2.3

So the simplified circuit has two inductances , two resistances, a capacitance, three independent initial conditions and therefore the characteristic equation will be a third-order one.

### 2.4. Determination of Integration Constants in the Classical Method

As stated earlier, any free component of transient current or voltage may be represented as the sum of exponential terms. The number of such terms is equal to the number of roots of the characteristic equation.

For any network one can find by Kirchhoff`s laws and the switching rules:

1. The numerical value of $i_L(-0)$, or the transient current at t=-0.
2. The numerical value of the first and if necessary, higher derivatives of the transient current for t=0.

According to (2.2) and (2.4) the transient current of k-th branch and higher derivatives of the transient current are

$$\left.\begin{aligned} & i_k = i_k^{'} + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + \dots\dots + A_n e^{\alpha_n t} \\ & \frac{di_k}{dt} = \frac{di_k^{`}}{dt} + A_1\alpha_1 e^{\alpha_1 t} + A\ \alpha_2 e^{\alpha_2 t} + \dots\dots + A_n e^{\alpha_n t} \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \frac{di_k^{n-1}}{dt^{n-1}} = \frac{d^{n-1} i_k^{`}}{dt^{n-1}} + A_1\alpha_1^{n-1} e^{\alpha_1 t} + A_2\alpha_2^{n-1} e^{\alpha_2 t} + \dots + A_n\alpha_n^{n-1} e^{\alpha_n t}. \end{aligned}\right\} \qquad 2.6$$

Using the numerical values of the transient current and the first and higher derivatives of the transient current at t=+0 the following set of equations may be written

$$\left.\begin{aligned} & i_k(-0) = i_k^{'}(+0) + A_1 + A_2 + \dots\dots + A_n \\ & \left[\frac{di_k}{dt}\right]_{t=-0} = \left[\frac{di_k^{'}}{dt}\right]_{t=+0} + A_1\alpha_1 + A_2\alpha_2 + \dots\dots A_n\alpha_n \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & \left[\frac{d^{n-1} i_k}{dt^{n-1}}\right]_{t=-0} = \left[\frac{d^{n-1} i_k}{dt^{n-1}}\right]_{t=+0} + A_1\alpha_1^{n-1} + A_2\alpha_2^{n-1} + \dots\dots + A_n\alpha_n^{n-1}. \end{aligned}\right\} \qquad 2.7$$

The set of equations (2.7) contains "n" number of equations and "n" number of unknown integration constants $A_1$, $A_2$, ……$A_n$. This set may be solved.

**2.5. The transient in R-L Circuit**

Suppose a coil is to be connected to the source of supply with driving voltage v(t). The circuit after switching is shown in Fig.2.4. The fundamental variable is “i”. Using Kirchhoff`s

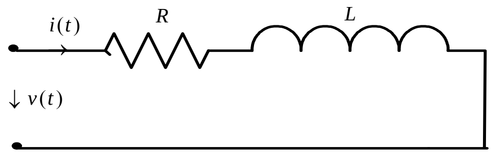



Fig.2.4

voltage law differential equation of the circuit will be

$$L\frac{di}{dt}+Ri=v(t) \qquad 2.8$$

This is the non-homogeneous differential equation of the first order. The solution of this equation is

$$i(t)=i^{'}(t)+Ae^{\alpha t}, \qquad 2.9$$

where $i^{'}(t)$ is the steady-state component of the transient current, A – is the integration constant, α is the root of the following characteristic equation

$$L\alpha+R=0, \quad \Rightarrow \alpha=-\frac{R}{L}. \qquad 2.10$$

Hence the transient current is

$$i(t)=i^{'}(t)+Ae^{-\frac{R}{L}t}. \qquad 2.11$$

The value L/R=τ has dimension of time. It is called as the time constant and equal to the time that is necessary the transient current to be changed e ≈ 2.71 times.

Let’s consider several cases.

a) A coil is to be connected to DC source of supply. Let’s fined transient current i(t) and voltages $v_R(t)$ and $v_L(t)$. According to the formula (2.11) the transient current

$$i(t)=\frac{V_0}{R}+Ae^{-\frac{R}{L}t}, \qquad 2.12$$

where the steady-state component of current $I^{`}(t)=V_0/R$ and according to the first switching rule $i_L(-0)=I_L(+0)$. Hence

$$0=\frac{V_0}{R}+Ae^{-\frac{0}{R}} \qquad \Rightarrow A=-\frac{V_0}{R}.$$

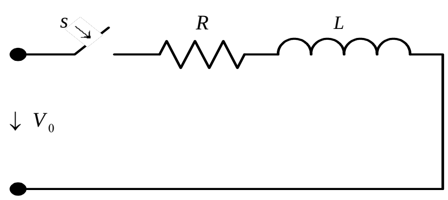



Fig.2.5

Finally, the transient current is

$$i(t)=\frac{V_0}{R}\left[1-e^{-\frac{t}{\tau}}\right]. \qquad 2.13$$

A t=0 the transient current is equal to zero. Then it increases according to exponential law and at t=∞ it will be equal to the steady-state component of transient current. The graph of the transient current is

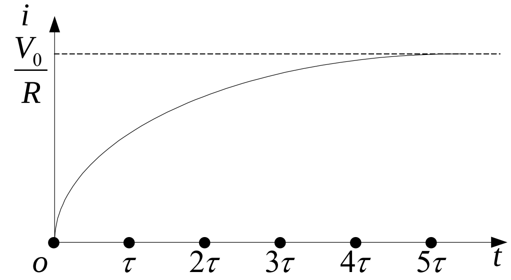


Fig.2.5

Theoretically transient period is infinite, but as it shown from Fig.2.5 practically the transient period is $(3 – 4)\tau$.

As to voltages $v_R$ and $v_L$ we have

$$V_R = Ri = V_0\left[1 - e^{-\frac{t}{\beta\tau}}\right]: \qquad V_L = L\frac{di}{dt} = V_0 e^{-\frac{t}{\tau}}. \qquad 2.14$$

Again transient voltages $v_R$ and $v_L$ are changed according to the exponential law. Their graphs are shown in Fig.2.6 The voltage across the inductance increases instantaneously from zero to $V_0$ at t=0. Then it decreases according to exponential law and becomes zero at $t=\infty$. As to the voltage $v_R$ it increases from zero to $V_0$ likewise the transient current.

The voltage across the inductance increases instantaneously from zero to $V_0$ at t=0. Then it decreases according to exponential law and becomes zero at $t = \infty$. As to the voltage $V_R$ it increases from zero to $V_0$ likewise the transient current.

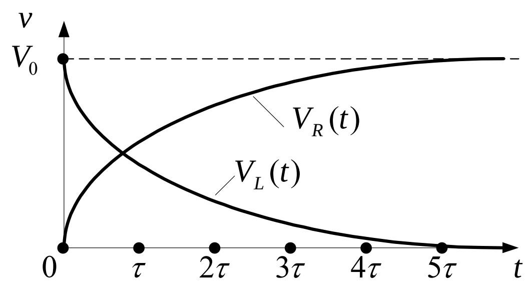


Fig.2.6

b). Suppose a coil is to be switched off from the DC source of supply. In order to avoid a sparking a resistor $R_0$ is used. After switching the circuit to be obtained looks as shown in Fig.2.7.

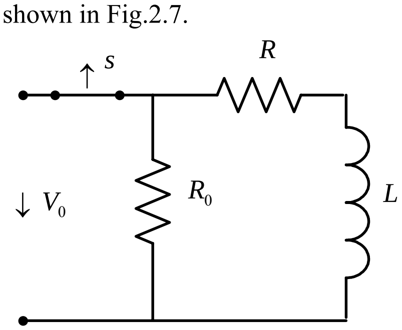


Fig.2.7

Corresponding differential equation is

$$L\frac{di}{dt} + (R + R_0)i = 0. \qquad 2.15$$

Because of the steady-state component of current is equal zero the transient current is

$$i = Ae^{-\frac{t}{\tau}}, \ \textit{where the time constant} \quad \tau = \frac{L}{R + R_0}.$$

In order to find the integration constant A we have to use the first switching rule $i_L(-0)=i_L(+0)$. Before switching

$$i_L = \frac{V_0}{R}.\ \ Therefore\ \ \frac{V_0}{R} = Ae^{-\frac{0}{\tau}} \quad \Rightarrow A = \frac{V_0}{R}$$

and the transient current is (Fig.2.8)

$$i(t) = \frac{V_0}{R} e^{-\frac{t}{\tau}}. \qquad 2.16$$

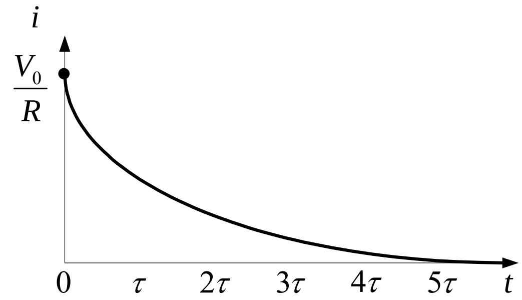



Fig.2.8

Very often it is supposed instead of $R_0$ to be a voltmeter. In this case voltage across the voltmeter is

$$v_{R_0}(t) = Ri(t) = \frac{R_0}{R} V_0 e^{-\frac{t}{\tau}}. \qquad 2.17$$

According to this formula voltage across to the voltmeter may be $R_0/R$ times more than the driving voltage. It must be taking into account due to avoid distortion of the instrument.

c). Suppose a coil having resistance R and inductance L is to be connected to AC voltage source with the sinusoidal voltage $v(t)=V_m\sin(\omega t+\psi_v)$. Then the transient current will be

$$i(t) = i'(t) + Ae^{-\frac{t}{\tau}}, \qquad 2.18$$

where the steady-state component of current

$$\left. \begin{aligned} i(t) &= \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t + \psi_v - \varphi) \\ \varphi &= tn^{-1} \frac{\omega L}{R}. \end{aligned} \right\} \qquad 2.19$$

Hence the transient current is

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t + \psi_v - \varphi) + Ae^{-\frac{t}{\tau}}. \qquad 2.20$$

Before switching $i_L(-0)=0$ and then according to (2.20)

$$0 = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\psi_v - \varphi) + A; \quad \Rightarrow A = -\frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\psi_v - \varphi). \qquad 2.21$$

Therefore the transient current is

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}} \sin(\omega t + \psi_v - \varphi) - \frac{V_m \sin(\psi_v - \varphi)}{\sqrt{R\ \ + (\omega L)^2}} e^{-\frac{t}{\tau}}. \qquad 2.22$$

The graph of the transient current versus time is shown in Fig 2.9

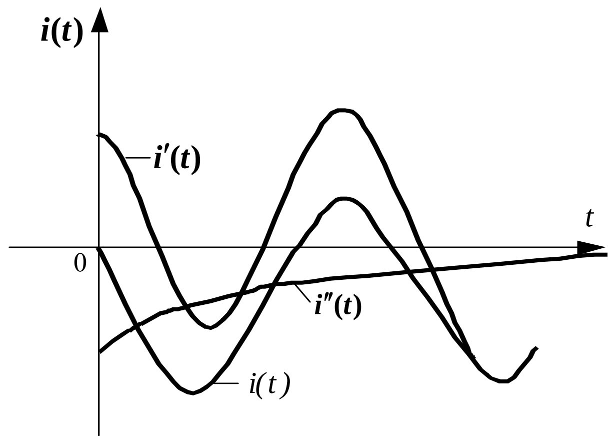



Fig.2.9

The graph is plotted for the case when $\psi_v - \varphi = \pi/2$.

Using the formula (2.22) the following conclusions may be done:

1. If the circuit parameters are chosen so that $\psi_v = \varphi$ the free component of transient current is equal to zero. In such circuit there is no transient performance at all.
2. When $\psi_v - \varphi = \pm\pi/2$ the free component of transient current is maximum.
3. The maximum value of transient current may be two times more than the peak value of steady-state component of current.

### 2.6. The Transient in R-C Circuit

Suppose a capacitor is to be connected to the source of supply with driving voltage v(t) as

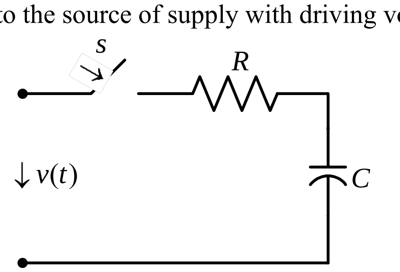



Fig.2.10

it is shown in Fig.2.10. Using Kirchhoff's voltage law the differential equation may be written as

$$RC\frac{dv_C}{dt} + v_C = v(t). \qquad \textbf{2.23}$$

In this equation the fundamental variable is the voltage across the capacitor. It is chosen because the voltages across the capacitor before and after switching are in relationship according to the second switching rule.

Solution of the equation (2.23) is

$$v_C(t) = v_C^{'}(t) + Ae^{\alpha t}, \qquad 2.24$$

where $v_C^{'}(t)$ - is the steady-state voltage,
A - is the integration constant,
$\alpha$ - is the root of the following characteristic equation

$$RC\alpha + 1 = 0; \qquad \Rightarrow \alpha = -\frac{1}{RC}.$$

Hence the transient voltage is

$$v_C(t) = v_C^{'}(t) + Ae^{-\frac{t}{RC}}. \qquad 2.25$$

The value RC=τ has dimension of time. It is called as the circuit time constant. Let's consider several cases.

a). A capacitor is to be connected to DC source of supply. After switching the capacitor starts to charged and its Steady-state voltage will be $v_C'(t)=V_0$ and according to the formula (2.25) the transient voltage is

$$v_C(t) = V_0 + Ae^{-\frac{t}{\tau}}. \qquad 2.26$$

Due to find the integration constant A let's use the second switching rule $v_C(-0)=v_C(+0)$.

Before switching $v_C(-0)=0$. Hence

$$o = V_0 + e^{-\frac{0}{\tau}}; \quad \Rightarrow \; A = -V_0.$$

Finally the transient voltage is

$$v_C(t) = V_0\left[1 - e^{-\frac{t}{\tau}}\right]. \qquad 2.27$$

As to the charging (transient) current it will be

$$i(t) = C\frac{dv_C}{dt} = \frac{V_0}{R}e^{-\frac{t}{\tau}} \qquad 2.28$$

and the transient voltage across resistance will be

$$v_R(t) = iR = V_0 e^{-\frac{t}{\tau}}. \qquad 2.29$$

The transient current and voltages in the case of charging the capacitor vary according to exponential law. Their graphs are shown in Fig.2.11.

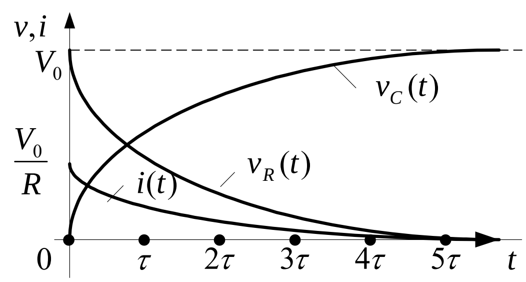



Fig.2.11

As it's clear from the above curves practically the transient period is 3 – 4 τ. The voltage across capacitance cannot change instantaneously, but current and voltage across resistance can.

Let's find the energy drawn by the circuit from the source of supply during the transient period.

$$W = \int_0^\infty V_0 i(t)dt = V_0^2\, C.$$

So it is exactly two times more then the energy stored in the electric field of the capacitor. It may be shown that the half of this energy is dissipated in resistance and the another half is stored in the electric field.

b). A capacitor is to be disconnected from DC source of supply

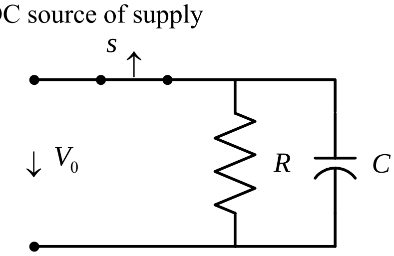



Fig.2.12

Actually, after switching discharge of the capacior takes place as it is shown in Fig.2.12 . According to Kirchhoff's voltage law may be written as $iR+v_C=0$ and than differential equation of the circuit will be

$$C\frac{dv_C}{dt}+v_C=0 \qquad 2.30$$

the solution of which is

$$v_C=Ae^{-\frac{t}{\tau}}. \qquad 2.31$$

The initial condition of this problem is $v_C(-0)=V_0$ and accordance to the eq. (2.31)

$$V_0=Ae^{-\frac{0}{\tau}}; \qquad \Rightarrow A=V_0$$

and the transient voltage is

$$v_C(t)=V_0e^{-\frac{t}{\tau}} \qquad 2.32$$

The transient current will be

$$i(t)=C\frac{dv_C}{dt}=-\frac{V_0}{R}e^{-\frac{t}{\tau}}. \qquad 2.33$$

The graphs of $v_C(t)$ and $i(t)$ versus time are shown in Fig.2.13.

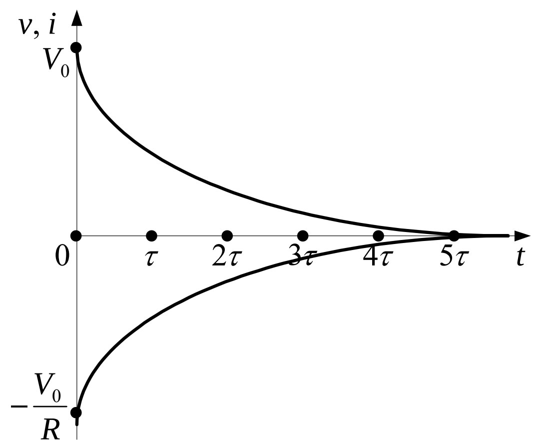



Fig.2.13

The energy stored in the electric field of the capacitor is $W_e=CV_0^2/2$ and it is completely dissipated in the resistor

$$W_R=\int_0^\infty i^2Rdt=\frac{V_0^2C}{2}.$$

c). A capacitor is to be connected to the AC source with sinusoidal voltage $v(t)=V_m\sin(\omega t+\psi_v)$. Then steady-state current is

$$i'(t)=\frac{V_m}{\sqrt{R^2+\left[\frac{1}{\omega C}\right]^2}}\sin(\omega t+\psi_v+\varphi)=I_m\sin(\omega t+\psi_v+\varphi) \qquad 2.34$$

and the steady-state voltage across capacitance is

$$v_C'(t)=\frac{1}{\omega C}I_m\sin(\omega t+\psi_v+\varphi-\frac{\pi}{2}). \qquad 2.35$$

Then the transient voltage is

$$v_C(t)=\frac{1}{\omega C}I_m\sin(\omega t+\psi_v+\varphi-\frac{\pi}{2})+Ae^{-\frac{t}{\tau}} \qquad 2.36$$

Before switching the voltage across capacitance $v_C(-0)=0$. Then integration constant is:

$$0=I_m\frac{1}{\omega C}\sin(\psi_v+\varphi-\frac{\pi}{2})+A; \ \Rightarrow A=-I_m\frac{1}{\omega C}\sin(\psi_v+\varphi-\frac{\pi}{2})$$

and the transient voltage

$$v_C(t) = \frac{1}{\omega C} I_m \sin(\omega t + \psi_v + \varphi - \frac{\pi}{2}) - \frac{1}{\omega C} I_m \sin(\psi_v + \varphi - \frac{\pi}{2}) e^{-\frac{t}{\tau}}. \qquad 2.37$$

Hence the transient current is

$$i(t) = C\frac{dv_C}{dt} = I_m \sin(\omega t + \psi_v + \varphi) + \frac{1}{\omega RC} I_m \sin(\psi_v + \varphi - \frac{\pi}{2}) e^{-\frac{t}{\tau}}. \qquad 2.38$$

Analyze the expression of the transient voltage it is possible to make the following conclusion:

1. If $\psi_v = -\varphi + \pi/2$ the free component of transient voltage is equal to zero and there is no transient in electric circuit.
2. If switching takes place at the moment when $\psi_v = \varphi$ the free component of transient voltage is maximum.
3. The maximum value of transient voltage may be twice more than the steady-state voltage across capacitance.

The graph of transient current versus time is shown in Fig.2.14

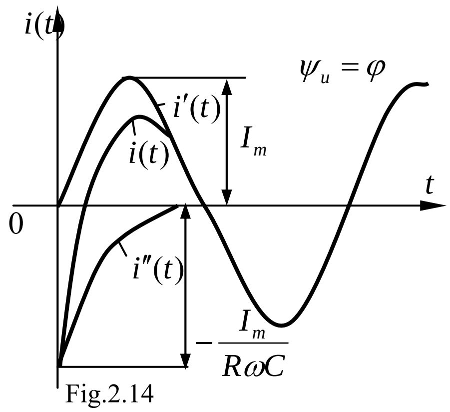



Fig.2.14

The graph is plotted for the case when $\psi_v + \varphi = 0$.

### 2.7. The Transient in R-L-C Circuit

It is supposed that a coil and a capacitor in series are connected to the source of supply as shown in Fig. 2.18.

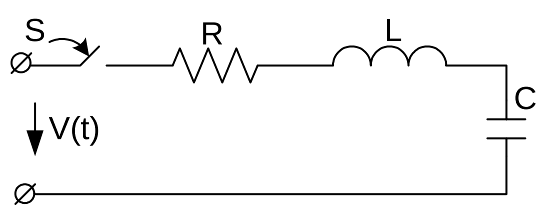



Fig. 2.18.

Using Kirchhoff ‘s voltage law the following equation may be written

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^1 idt + V_c(0) = V(t). \qquad 2.39$$

This is an integro-differential equation. Due to make it differential equation let‘s take for the derivative of this equation with respect to time.

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}. \qquad 2.40$$

(2.40) is the differential nonhomogeneous equation of the second order. Corresponding homogeneous equation is

$$L\frac{d^2 i''}{dt^2} + R\frac{di''}{dt} + \frac{i''}{C} = 0.$$

Divide this equation by L and using notations

$$\frac{R}{L} = 2\delta; \quad \frac{1}{LC} = \omega_0^2 . \qquad 2.41$$

we have

$$\frac{d^2 i''}{dt^2} + 2\delta \frac{di''}{dt} + \omega_0^2 i'' = 0 . \qquad 2.42$$

The characteristic equation is

$$\alpha^2 + 2\delta\alpha + \omega_0^2 = 0 . \qquad 2.43$$

This equation has two roots

$$\alpha_1 = -\delta + \sqrt{\delta^2 - \omega_0^2}; \quad \alpha_2 = -\delta - \sqrt{\delta^2 - \omega_0^2} . \qquad 2.44$$

There may be three cases:

a) two unequal negative real roots
b) two equal negative real roots
c) two conjugate complex roots with negative real part..

The solution of the equation (9.42) is

$$i''(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$$

and the transient current is

$$i(t) = i'(t) + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \quad . \qquad 2.45$$

where $i'(t)$ is the steady-state component and $A_1$, $A_2$ are the integration constants. Due to find $A_1$ and $A_2$ we have to use the following set of equations

$$\left.\begin{aligned} i(t) &= i'(t) + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \\ \frac{di}{dt} &= \frac{di'}{dt} + A_1 \alpha_1 e^{\alpha_1 t} + A_2 \alpha_2 e^{\alpha_2 t} \end{aligned}\right\} \qquad 2.46$$

This set of equations at the switching moment (t=0) may be written as

$$\left.\begin{aligned} i(0) &= i'(0) + A_1 + A_2 \\ \frac{di}{dt_{t=0}} &= \frac{di'}{dt_{t=0}} + A_1\alpha_1 + A_2\alpha_2 \end{aligned}\right\} \qquad 2.47$$

Using switching rules and the equation (9.39) it's possible to write

$$L\left(\frac{di}{dt}\right)_{t=0} + Ri(0) + v_C(0) = v(0) \Rightarrow \quad \frac{di}{dt_{t=0}} = \frac{v(0) - Ri(0) - v_C(0)}{L}$$

Hence instead the set of equations (2.47) we have

$$\left.\begin{aligned} i(0) &= i'(0) + A_1 + A_2 \\ \frac{v(0) - Ri(0) - v_C(0)}{L} &= \frac{di'}{dt_{t=0}} + A_1\alpha_1 + A_2\alpha_2 \end{aligned}\right\} \qquad 2.48$$

This set of equations allows to find the constants of integration $A_1$ and $A_2$.

## 2.8. Discharge of a Capacitor by a Coil

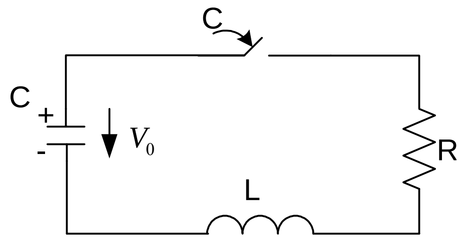



Fig. 2.19

It is supposed that a charged capacitor with an initial voltage $v_C(-0)=V_0$ after switching operation is discharged by a coil with R-L parameters. There is no steady state component and the transient current is $i(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t}$. In this case i'(0)=0, $\frac{di'}{dt_{t=0}} = 0$. the initial current is i(-0)=i(0)=0 and $\frac{di}{dt_{t=r}} = \frac{v(0) - Ri(0) - v_C(0)}{L} = -\frac{V_0}{L}$.

Using these initial conditions the set (9.48) may be written as

$$0 = A_1 + A_2 \qquad \Rightarrow \quad A_1 = -A_2$$

$$-\frac{V_0}{L} = A_1\alpha_1 + A_2\alpha_2 \Rightarrow A_1 = -A_2 = -\frac{V_0}{L(\alpha_1 - \alpha_2)}$$

Hence the transient current will be

$$i(t) = -\frac{V_0}{L(\alpha_1 - \alpha_2)}(e^{\alpha_1 t} - e^{\alpha_2 t}), \qquad 2.49$$

The transient voltages of capacitance and inductance are

$$V_C = \frac{1}{C}\int_0^1 idt + V_0 = -\frac{V_0}{\alpha_1 - \alpha_2}(\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}), \qquad 2.50$$

$$V_L = L\frac{di}{dt} = -\frac{V_0}{\alpha_1 - \alpha_2}(\alpha_1 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}). \qquad 2.51$$

The equations (2.50) and (2.51) may be derived taking into account that

$$\alpha_1\alpha_2 = \omega_0^2 = \frac{1}{LC}$$

As it has been mentioned previously the transient process depends on the value of roots of characteristic equation. Let's consider the following three cases.

a) The roots are negative and unequal values. This case takes place when $\delta > \omega_0$ or $\frac{R}{2l} > \frac{1}{\sqrt{LC}}$ or $R > 2\rho$. Here $\rho = \sqrt{\frac{L}{C}}$ is the wave resistance of the circuit

Because of $\alpha_1 < 0$ and $\alpha_2 < 0$ and $|\alpha_2| > |\alpha_1|$ when "t" changes from 0 to $\infty$ the values $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$ decrease from 1 to 0. It must be noted that $e^{\alpha_1 t}$ and $e^{\alpha_2 t}$ are always positive and $e^{\alpha_2 t}$ decreases faster than $e^{\alpha_1 t}$. The curves of $e^{\alpha_1 t}$, $e^{\alpha_2 t}$ and $e^{\alpha_1 t}$ - $e^{\alpha_2 t}$ are shown in Fig. (2.20).

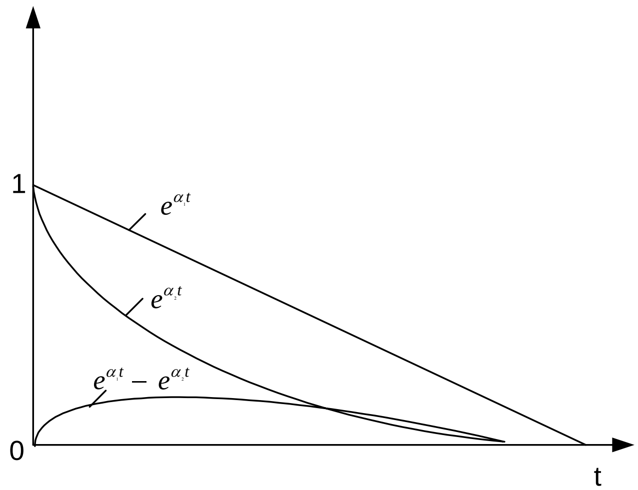



Fig. 2.20

The transient current curve has the same form as the function $e^{\alpha_1 t}$ - $e^{\alpha_2 t}$. The graph of the transient current is shown in Fig.(2.21). During discharge of capacitor the transient current never changes its direction. It increases from 0 to $I_{max}$ at=$t_m$. Then it decreases to zero at $t = \infty$.

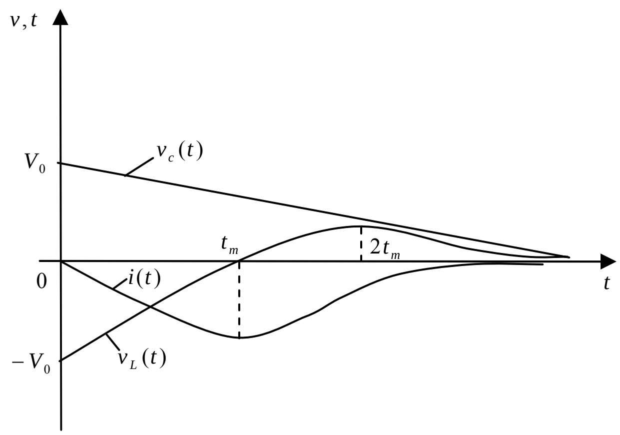



Fig. 2.21

The voltage across resistance R is

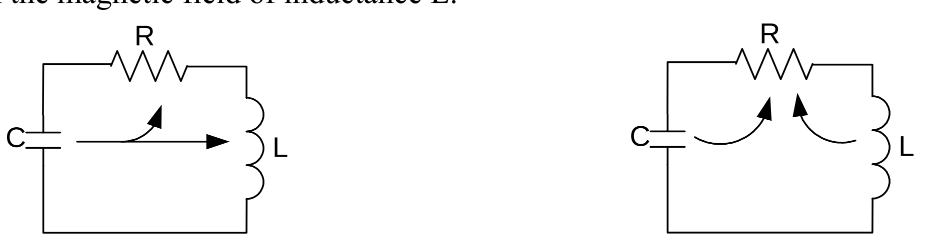
$$v_R(t) = Ri = -\frac{RV_0}{L(\alpha_1 - \alpha_2)}\left(e^{\alpha_1 t} - e\alpha_2 t\right) \qquad 2.52$$

and it repeats the wave form of current.

The voltage $v_L$= -$V_0$ at t=0.

This performance is called as nonperiodical discharge of capacitor.

At the time interval 0<t<$t_m$ the voltage across capacitance is positive $v_C$>0, but the current is negative i<0 and the power $p_C$=$v_C$ i<0. It means that the circuit draws energy from electric field of the capacitor. The part of this energy is dissipated as heat in resistance R and another part of energy is stored in the magnetic field of inductance L.




0 < t < $t_m$      $t_m$ < t < ∞

a)      b)

Fig. 2.22

At the time interval $0 < t < t_m$ the energy drawn from the electric field of capacitor is dissipated as heat in resistance. In this time interval energy is also drawn from the magnetic field of inductance and is dissipated as heat in resistance R. These processes are illustrated in Fig. (2.22 a and b).

the time "$t_m$" at which current reaches its maximum may be found by the following equation.

$$\frac{di}{dt}=0;\ -\frac{V_0}{L(\alpha_1-\alpha_2)}\left(\alpha_1 e^{\alpha_{11}t}-\alpha_2 e^{\alpha_2 t}\right)=0$$

$$\alpha_1 e^{\alpha_1 t_m}-\alpha_2 e^{\alpha_2 t_m}=0;\ \ \frac{\alpha_1}{\alpha_2}=\frac{e^{\alpha_2 t_m}}{e^{\alpha_1 t_m}}=e^{(\alpha_2-\alpha_1)t_m}$$

$$In\frac{\alpha_1}{\alpha_2}=(\alpha_2-\alpha_1)t_m \Rightarrow \quad t_m\frac{In\frac{\alpha_1}{\alpha_2}}{\alpha_2-\alpha_1}$$

b) The roots are negative equal values. This case takes place when $\delta=\omega_0$ that means the resistance $R=2\sqrt{\frac{L}{C}}=2\rho.$ So the roots $\alpha_1=\alpha_2=-\delta$.

Because there are zeros in denominators of formulas (2.49), (2.50) and (2.51) transient current and voltages are found as.

$$i(t)=-\frac{V_0}{L}\lim_{\alpha_1\to\alpha_2}\frac{d}{d\alpha_1}\frac{e^{\alpha_1 t}-e^{\alpha_2 t}}{\alpha_1-\alpha_2}=-\frac{V_0}{L}\lim_{\alpha_1\to\alpha_2}\frac{te^{\alpha_1 t}}{1}=-\frac{V_0}{L}te^{\alpha_2 t}=-\frac{V_0}{L}te^{-\delta t} \qquad 2.53$$

$$v_C=-V_0\lim_{\alpha_1\to\alpha_2}\frac{d}{d\alpha_1}\frac{\alpha_2 e^{\alpha_1 t}-\alpha_1 e^{\alpha_2 t}}{\alpha_1-\alpha_2}=-V_0\lim_{\alpha_1\to\alpha_2}\frac{\alpha_2 te^{\alpha_1 t}-e^{\alpha_2 t}}{1}=-V_0\frac{(\alpha_2 t-1)e^{\alpha_2 t}}{1}=V_0 e^{-\delta t}(1+\delta t) \qquad 2.54$$

$$v_L=-V_0\lim_{\alpha_1\to\alpha_2}\frac{d}{d\alpha_1}\frac{\alpha_1 e^{\alpha_1 t}-\alpha_2 e^{\alpha_2 t}}{\alpha_1-\alpha_2}=-V_0\lim_{\alpha_1\to\alpha_{21}}\frac{e^{\alpha_1 t}-\alpha_1 te^{\alpha_1 t}}{1}=-V_0 e^{-\delta t}(1+\delta t). \qquad (9.55)$$

Waveforms of current and voltages are like those in the case of "a" when the roots are negative different values. The current reaches maximum at time $t=t_m=\frac{1}{\delta}$.

c) The roots are complex conjugate numbers. This case takes place when $R<2\rho$. Let's introduce the notation $\omega'=\sqrt{\omega_0^2-\delta^2}$. Then the roots may be written as

$$\alpha_1=-\delta+\sqrt{\delta^2-\omega_0^2}=-\delta+j\omega=\omega_0 e^{j(\pi-\zeta)}$$
$$\alpha_2=-\delta-\sqrt{\delta^2-\omega_0^2}=-\delta-j\omega=\omega_0 e^{j(\pi-\zeta)}$$

where $\zeta=tn^{-1}\frac{\omega}{\delta}$.

For the transient current we have the following expression

$$i(t)=-\frac{V_0}{L(\alpha_1-\alpha_2)}(e^{\alpha t}-e^{\alpha_2 t})=-\frac{V_0}{2j\omega' L}\left(e^{-\delta t}e^{j\omega t}-e^{-\delta t}e^{-j\omega t}\right)=$$
$$=-\frac{V_0}{2j\omega' L}e^{-\delta t}\left(\cos\omega' t+j\sin\omega' t-\cos\omega' t+j\sin\omega' t\right)=-\frac{V_0}{\omega' L}e^{-\delta t}\sin\omega' t \qquad 2.56$$

As to transient voltage $v_L$ we have

$$v_L = -\frac{V_0}{\alpha_1 - \alpha_2}\left(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t}\right) = -\frac{V_0}{2j\omega'}\left[\omega_0 e^{j\pi} e^{-\delta t} e^{j(\omega' t - \varsigma)} - \omega_0 e^{j\pi} e^{-\delta t} e^{-j(\omega' t - \varsigma)}\right] =$$
$$= V_0 \frac{\omega_0}{\omega'} e^{-\delta t} \sin(\omega' t - \zeta) \qquad 2.57$$

The transient current and voltages are decreased according to damped sinusoidal function. The peak value of their function decrease according to exponential law. The graph of current is shown in Fig (9.23).

The value $\omega_0$ is called as angular frequency of natural or undamped oscillation of current or voltages. The value $\omega'$ in called as angular frequency of damped oscillation of current or voltages. The value

$$T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\omega_0^2 - \delta^2}} = \frac{2\pi}{\sqrt{1/LC - R^2/4L^2}}$$

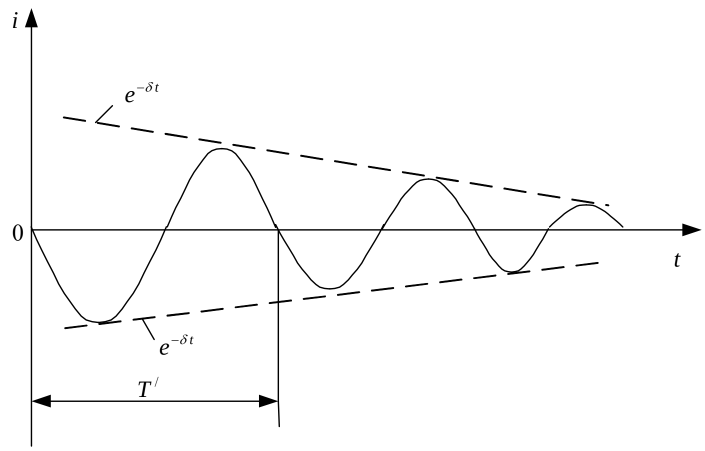



Fig.2.23

is called the period of damped oscillation. When $R = 0$ $T' = T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{LC}$ is the period of undamped oscillation. For this case the graph of i(t) and $v_C$(t) are shown in Fig 9.24

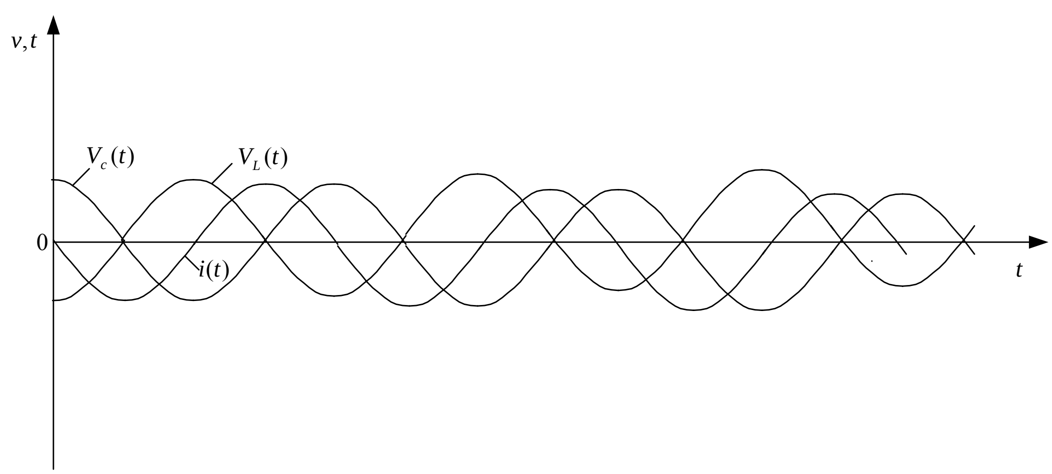



Fig 2.24

The rate of decrease of damped oscillation of current or voltages is evaluated by the decrement of oscillation

$$\Delta = I_m e^{-\delta t} \Big/ I_m e^{-\delta(t+T')} = e^{\delta T'} \qquad 2.58$$

where $I_m = \dfrac{V_0}{\omega' L}$,

or by the logarithmic decrement of oscillation.

$$\zeta = \ln\Delta = \delta T'$$

If the rate of decrease of amplitude is not very high then $T' \approx T_0$ and

$$\zeta = \delta T_0 = \frac{R}{2L} 2\pi\sqrt{LC} = \frac{\pi R}{\rho} = \frac{\pi}{Q}, \qquad 2.59$$

where Q is the quality factor.

### 2.9. The Transient in R-L-C Circuit at Switching it to DC Source of Supply

Suppose a coil and a capacitor connected in series are switched on to the DC source of supply with the voltage $V_0$ as shown in Fig.(2.25).

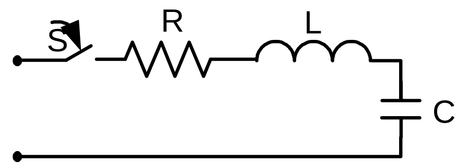



Fig. 2.25

It is supposed that the initial current and the initial voltage of capacitor i(-0)=0, $v_C(-0)=0$. The network equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i dt + v_C(0) = V_0, \qquad 2.60$$

The solution of this equation _ the transient current

$$i(t) = i(t) + A_1 e^{\alpha_1 t} + A_2 e^{2t}\ , \qquad 2.61$$

In this case the steady-state component of transient current

$i'(t) = 0$ and $i(t) = A_1 e^{\alpha_1 t} + A_2 e^{2t}$

From the equation (2.60)

$$\frac{di}{dt_{t=0}} = \frac{V_0 - Ri(0) - v_C(0)}{L}$$

At the given initial conditions

$$\frac{di}{dt_{t=0}} = \frac{V_0}{L}$$

and the set of equation for determination of the integrating constants is

$$\left.\begin{aligned} 0 &= A_1 + A_2 \\ \frac{V_0}{L} &= A_1\alpha_1 + A_2\alpha_2 \end{aligned}\right\} \qquad 2.62$$

from this set of equations the integrating constants are

$$A_1 = -A_2 = \frac{V_0}{L(\alpha_1 - \alpha_2)}$$

The transient current and voltages are

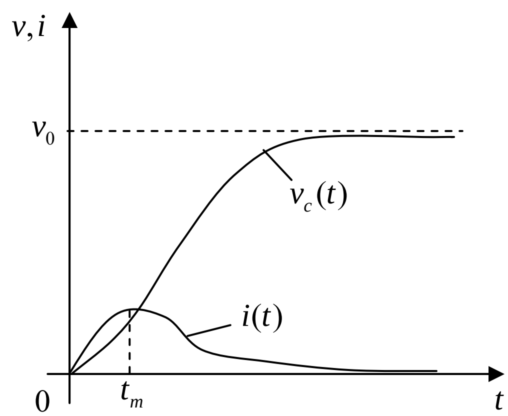
$$i(t) = \frac{V_0}{L(\alpha_1 - \alpha_2)}(e^{\alpha_1 t} - e^{\alpha_2 t}) \qquad 2.63$$

$$v_L(t) = \frac{V_0}{\alpha_1 - \alpha_2}(\alpha_1 e^{\alpha_1 t} - \alpha_2 e^{\alpha_2 t}) \qquad 2.64$$

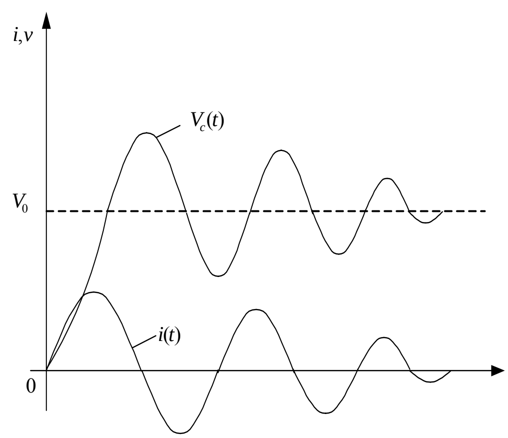
$$v_C(t) = \frac{V_0}{\alpha_1 - \alpha_2}(\alpha_2 e^{\alpha_1 t} - \alpha_1 e^{\alpha_2 t}) + V_0 \qquad 2.65$$

If we compare the formulas (2.63), (2.64) and (2.65) to the corresponding formulas for the case of discharge of a capacitor by a coil it is not difficult to note that the difference is only in sign.

The charging process of the capacitor depends on the relationship between circuit parameters. When the roots of characteristic equation are negative real values the process is nonperiodical and curves of current and $v_C$ voltage are shown in Fig. (2.26 a). In the case when the roots are complex number the charging process looks like oscillation and the curves of




a)




b)

Fig.2.26

current and $v_C$ voltage are shown in Fig. (2.26 b).

## 3. The Laplace Transform Method

### 3.1. Introduction to the Laplace Transform Method

From a course of mathematics it is known that logarithms allow to simplify the manipulation of numbers containing many decimal places. Thus, multiplication is reduced to adding and division to subtracting.

Since each number has a logarithm of its own, we may say that a logarithm is a transform of a definite number. For example 0.30103 is the transform (logarithm) of 2 to base 10.

Another example of introduction of the concept of transforms was in network analysis the complex numbers. A complex harmonic amplitude may be regarded as a transform of a sinusoidal function. Thus $\dot{I}_m$ represents a sinusoidal current $I_m \sin(\omega t + \psi)$. As logarithms simplify operations on numbers, the complex numbers simplify the manipulation of sinusoidal functions.

The Laplace transform method (LTM) that we are going to take up is based on the concept of time-function transforms of a function of complex variable $p = \delta + j\eta$.

The conversion of a time function into the respective "p" function is done by means of the LT, so the method on it is termed the LTM.

This method allows to reduce differentiation to multiplication and integration to division, thereby simplifying of differential equations. So this method allows instead differential equations solve corresponding algebraic equations.

The LTM is based on the following integral:

$$F(p) = \int_0^\infty f(t) e^{-pt} dt, \qquad 3.1$$

where $f(t)$ is the time function and F(p) is its Laplace transform. The correspondence between them is written as

$$F(p) \Leftrightarrow f(t),$$

where $\Leftrightarrow$ the sing of correspondence.

The time function may be considered as the inverse transform of F(p) and may be found by the following integral:

$$f(t) = \frac{1}{2\pi j} \int_{\delta - j\infty}^{\delta + \infty} F(p) e^{pt} dp. \qquad 3.2$$

It's known from mathematics that the integral (3.1) converges if and only if the function f(t) grows in magnitude with time more slowly, than the absolute value of the function $e^{\delta t}$. Practically all time functions f(t) of interest to electrical engineering satisfy this condition.

### 3.2. The LT of a Constant and an Exponential Function

Suppose f(t)=A, where A is a constant value. Substituting A for f(t) in (3.1) and integrating we have

$$F(p) = \int_0^\infty A e^{-pt} dt = A\left(-\frac{1}{p}\right) e^{-pt}\Big|_0^\infty = -\frac{A}{p}(0-1) = \frac{A}{p}.$$

Thus

$$A \Leftrightarrow \frac{A}{p} \quad . \qquad 3.3$$

If $f(t) = e^{\delta t}$ then the LT of the exponential function is

$$F(p) = \int_0^\infty e^{\alpha t} e^{-pt} dt = \int_0^\infty e^{-(p-\alpha)t} dt = -\frac{1}{p-\alpha} e^{-(p-\alpha)}\Big|_0^\infty = \frac{1}{p-\alpha}.$$

Thus

$$e^{\alpha t} \Leftrightarrow \frac{1}{p-\alpha} \; . \qquad 3.4$$

The transform of the exponential function $e^{-\alpha t}$ is

$$e^{-\alpha t} \Leftrightarrow \frac{1}{p+\alpha} . \qquad 3.5$$

If $\alpha = j(\omega t + \psi)$

$$e^{j(\omega t+\psi)} \Leftrightarrow e^{j\psi} \frac{1}{p - j\omega} . \qquad 3.6$$

## 3.3. The LT of a First and High Order Derivatives

Suppose F(p) is the transform of f(t). We inquire about the transform of a first derivative $\frac{df}{dt}$, knowing that for t=0 f(t)=f(0). Using (3.1) we have

$$\frac{df}{dt} = f' \Leftrightarrow \int_0^\infty f'(t) e^{-pt} dt .$$

Assume that $u = e^{-pt}$ $dv = f'(t)dt$. Then $du = -pe^{-pt}dt$ and $dv = f(t)$.
Integration by parts gives

$$\int udv = uv - \int vdu ,$$

$$\int_0^\infty f'(t) e^{-pt} dt = e^{-pt} f(t)\Big|_0^\infty - \int_0^\infty f(t)\left(-pe^{p-t}\right)dt = -f(0) + pF(p). \qquad 3.7$$

Therefore the transform of the first derivative is

$$f'(t) \Leftrightarrow pF(p) - f(0). \qquad 3.8$$

The LT of the second derivative is

$$f''(t) \Leftrightarrow p[pf(p) - f(0)] - f'(0) = p^2\left[F(p) - \frac{f(0)}{p} - \frac{f'(0)}{p^2}\right]. \qquad 3.9$$

The LT of the derivative of high order is

$$f^{(n)}(t) \Leftrightarrow p^n\left[F(p) - \frac{f(0)}{p} - \frac{f'(0)}{p^2} - \cdots - \frac{f^{(n-1)}(0)}{p^n}\right]. \qquad 3.10$$

In the case of zero initial conditions

$$f^{(n)}(t) \Leftrightarrow p^n F(p) . \qquad 3.11$$

### 3.4. The LT of an Integral

We inquire the LT of a function $\psi(t) = \int_0^t f(t)dt$, knowing that the LT of function $f(t)$ is $F(p)$.

$$\psi(t) \Leftrightarrow \int_0^\infty \psi(t) e^{-pt} dt .$$

Assume that $u = \psi(t);\; du = f(t)dt;\; dv = e^{-pt}dt;\; v = -\frac{1}{p}e^{-pt}$. Integrating by parts gives

$$\int udv = uv - \int vdu$$

$$\int_0^\infty \psi(t)e^{-pt}dt = \psi(t)\left(-\frac{1}{p}e^{-pt}\right)\Big|_0^\infty - \int_0^\infty \left(-\frac{1}{p}e^{-pt}\right)f(t)dt = \frac{1}{p}F(p). \qquad 3.12$$

Hence in order to find the LT integral of a function it is necessary the LT of function divide by the operator p.

### 3.5. The LT of Some functions

Suppose $f(t) = \sin\omega t.$ The LT of sinusoidal function will be

$$\sin\omega t = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \Leftrightarrow \frac{1}{2j}\left[\frac{1}{p - j\omega} - \frac{1}{p + j\omega}\right] = \frac{1}{2j}\frac{p + j\omega - p + j\omega}{p^2 + \omega^2} = \frac{\omega}{p^2 + \omega^2}.$$

So

$$\sin\omega t \Leftrightarrow \frac{\omega}{p^2 + \omega^2}. \qquad 3.13$$

Similarly the LT of cosine function is

$$\cos\omega t = \frac{1}{2}\left(e^{j\omega t} - e^{-j\omega t}\right) \Leftrightarrow \frac{1}{2j}\left[\frac{1}{p - j\omega} + \frac{1}{p + j\omega}\right] = \frac{p}{p^2 + \omega^2}.$$

Therefore

$$\cos\omega t \Leftrightarrow \frac{p}{p^2 + \omega^2}. \qquad 3.14$$

The LT of the function

$$\sin(\omega t + \psi) \Leftrightarrow \frac{\omega\cos\psi + p\sin\psi}{p^2 + \omega^2}; \qquad 3.15$$

$$e^{-\delta t}\sin\omega t \Leftrightarrow \frac{\omega}{(p + \delta)^2 + \omega^2}; \qquad 3.16$$

$$te^{-\delta t} \Leftrightarrow \frac{1}{(p + \delta)^2}; \qquad 3.17$$

$$t \Leftrightarrow \frac{1}{p^2}. \qquad 3.18$$

### 3.6. Ohm's and Kirchhoff's Laws in Operational Form

Fig. 3.1 shows a branch of an electric circuit between the nodes “a” and “b” containing R, L and C.

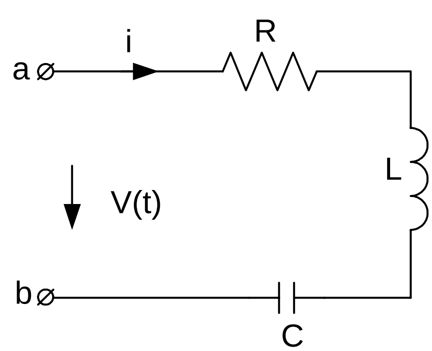



Fig. 3.1

The network equation for instantaneous values of voltage and current is

$$v(t) = Ri + L\frac{di}{dt} + \frac{1}{c}\int_0^t idt + v_c(0)\,. \qquad 3.19)$$

Rewrite this equation in operational form using the LT of a time functions v(t), i(t), a first derivative of time function, an integral and constant $v_c(0)$.

$$V(p) = RI(p) + LpI(p) - Li(0) + \frac{1}{Cp}I(p) + \frac{v_C(0)}{P}\,.$$

This equation may be written as

$$V(p) = I(p)Z(p) + \frac{v_C(0)}{P} - Li(0), \qquad 3.20$$

where

$$Z(p) = R + Lp + \frac{1}{Cp} \qquad 3.21$$

is the impedance in operational form or the LT impedance. Then from the equation (10.20) the LT current

$$I(p) = \frac{V(p) + Li(0) - \dfrac{v_C(0)}{p}}{Z(p)}\,. \qquad 3.22$$

This formula expresses Ohm's law in operation form. At zero initial conditions when i(0)=0 and $v_C(0) = 0$ Ohm's law is

$$I(p) = \frac{V(p)}{Z(p)}\,. \qquad 3.23$$

By Kirchhoff's current law for the node in the circuit

$$\sum_{k=1}^{n} i_k = 0\,.$$

Applying the LT to this equation and noting that the transform of functions is the sum of the transforms we get

$$\sum_{k=1}^{n} I_k(p) = 0 \qquad 3.24$$

This equation expresses Kirchhoff's current law in operational form.

Due to write equation according to Kirchhoff's voltage law in operational form let's consider a loop of a network shown in Fig. 3.2.

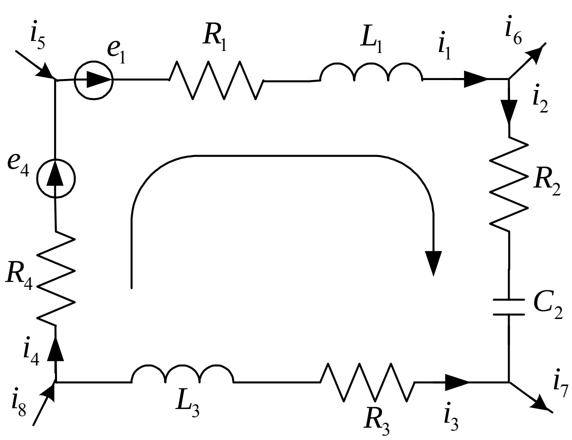



Fig. 3.2

Using direction of summation in Fig. 3.2 the equation for instantaneous values of currents, voltages and emfs is

$$e_1 + e_4 = R_1 i_1 + L_1\frac{di_1}{dt} + R_2 i_2 + \frac{1}{C_2}\int_0^t i_2 dt + v_{C_2}(0) - i_3 R_3 - L_3\frac{di_3}{dt} + R_4 i_4\,.$$

Let's write this equation in operational from

$$E_1(p)+E_4(p)=R_1I_1(p)+pL_1I_1(p)-L_1i_1(0)+R_2I_2(p)+\frac{1}{C_2p}I_2(p)+\frac{v_{C_2}(0)}{P}-$$
$$-I_3(p)R_3-pL_3I_3(p)+L_3i_3(0)+R_4I_4(p). \quad 3.24$$

In general case we have

$$\sum_{k=1}^{n}E_k(p)=\sum_{k=1}^{n}U_k(p), \quad 3.25$$

where $U_k(p)$ is the LT of the voltage across of k-th branch.

The equation (3.24) may be written as follows

$$E_1(p)+E_4(p)+L_1i_1(0)-\frac{v_C(0)}{P}-L_3i_3(0)=I_1(p)Z_1(p)+I_2(p)Z_2(p)-$$
$$-I_3(p)Z_3(p)+I_4(p)Z_4(p), \quad 3.26$$

where

$$Z_1(p)=R_1+L_1p;$$
$$Z_2(p)=R_2+\frac{1}{Cp};$$
$$Z_3(p)=R_3+L_3p;$$
$$Z_4(p)=R_{4.}$$

$Z_1(p), Z_2(p), Z_3(p), Z_4(p)$ are LT impedances of the branches. The values $L_1i_1(0), L_3i_3(0), \frac{v_C(0)}{P}$ may be considered as the supplementary e.m.f.s – as shown in Fig. 3.3.

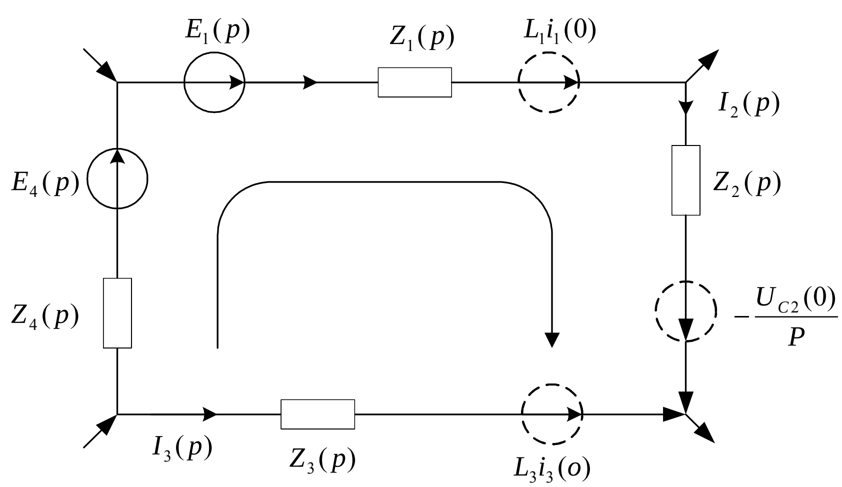



Fig. 3.3.

They allow to take into account nonzero initial conditions. Namely the term Li(0) is the supplementary emf due to the energy stored by the magnetic field of L and the term $\frac{v_C(0)}{P}$ is the supplementary emf due to the energy stored by the electric field of C.

**3. 5. The Transient Analysis of Networks by the LTM**

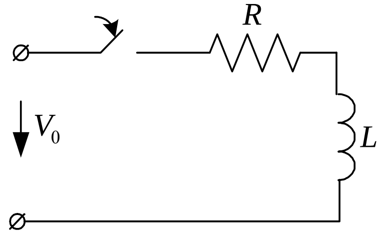



Fig. 3.4

Suppose a circuit shown in Fig. 3.4 includes a switch "s". At the zero initial conditions i(0)=0 it's necessary to find the transient current i(t). According to Ohm's law at zero initial conditions we have

$$I(p)=\frac{V(p)}{Z(p)}.$$

The LT impedance of the circuit is

$$Z(p)=R+Lp$$

The LT of the applied voltage is

$$V(p)=\frac{V_0}{P}.$$

Therefore the LT of current will be

$$I(p)=\frac{V_0}{p(R+L_P)}=\frac{V_0}{R}\left(\frac{1}{P}-\frac{1}{P+\frac{R}{L}}\right).$$

Corresponding time – function - the transient current will be

$$i(t)=\frac{V_0}{R}\left(1-e^{-\frac{R}{L}t}\right).$$

In the case of switching on R – C circuit to DC source of supply of driving voltage $V_0$ at zero initial condition the LT of transient current will be

$$I(p)=\frac{V(p)}{Z(p)},$$
$$Z(p)=R+\frac{1}{Cp},$$
$$V(p)=\frac{V_0}{P};\ I(p)=\frac{V_0}{p\left(R+\frac{1}{Cp}\right)}=\frac{V_0}{R}\frac{1}{p+\frac{1}{CR}}.$$

Corresponding transient current is

$$i(t)=\frac{V_0}{R}e^{-\frac{t}{RC}}.$$

In the case of switching R-L -C circuit to DC source of supply at zero initial conditions we have

$$I(p)=\frac{V(p)}{Z(p)};\ V(p)=\frac{V_0}{p};\ Z(p)=R+Lp+\frac{1}{Cp};$$

$$I(p)=\frac{V_0}{p\left(R+Lp+\frac{1}{Cp}\right)}=\frac{V_0}{L}\frac{1}{p^2+\frac{R}{L}p+\frac{1}{LC}}=\frac{V_0}{L}\frac{1}{\left(p+\frac{R}{2L}\right)^2+\frac{1}{LC}-\left(\frac{R}{2L}\right)^2}=\frac{V_0}{\omega' L}\frac{\omega'}{(p+\sigma)^2+\omega'^2}$$

where

$$\sigma = \frac{R}{2L}; \quad \omega_0^2 = \frac{1}{LC}; \quad \omega'^2 = \omega_0^2 - \sigma^2 .$$

The corresponding transient current is

$$i(t) = \frac{V_0}{\omega' L} e^{-\sigma t} \sin \omega' t.$$

Now let's consider the transient in two-mesh circuit shown in Fig. 3.5 in the case of nonzero

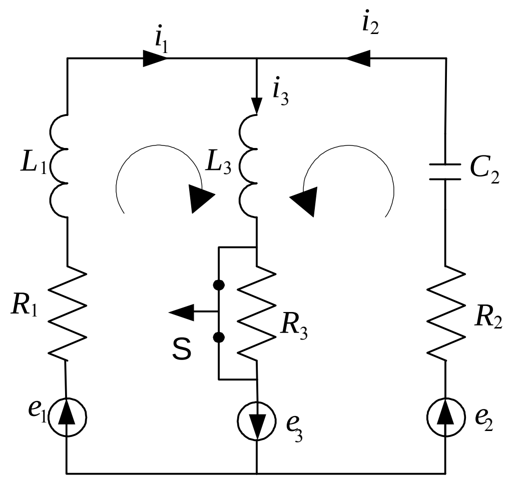



Fig. 3.5.

initial conditions when the values $i_1(0)$, $i_3(0)$ and $v_{C_2}(0)$ differ from zero. Let's take them into account by the supplementary energy sources If include them in the network we 'll have the circuit diagram shown in Fig.3.6.

Using the mesh current method the set of network equation is

$$\left.\begin{aligned} I_1(p)Z_{11}(p) + I_2(p)Z_{12}(p) = E_{11}(p) \\ I_1(p)Z_{21}(p) + I_2(p)Z_{22}(p) = E_{22}(p) \end{aligned}\right\} , \qquad 3.27$$

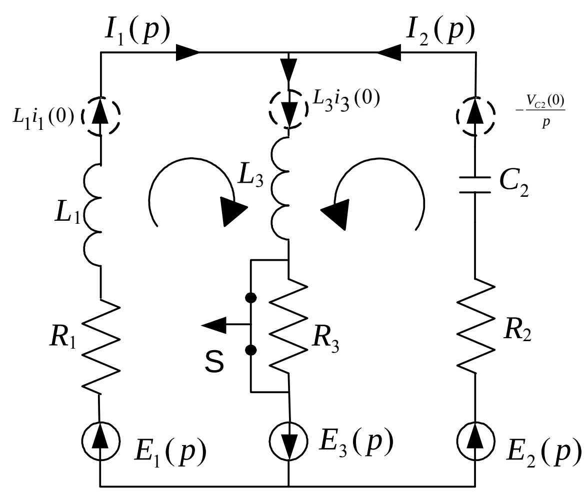



Fig. 3.6.

where $I_1(p)$ and $I_2(p)$ are LT of mesh currents $i_1(t)$ and $i_2(t)$.

$Z_{11}(p) = L_{1p} + R_1 + R_3 + L_{3p}$ = the LT self impedance of the first mesh.

$Z_{22}(p) = R_2 + \frac{1}{C_2 p} + R_3 + L_{3p}$ = the LT self impedance of the second mesh.

$Z_{12}(p) = Z_{21}(p) = R_3 + L_{3p}$ = the LT mutual impedance of the first and second meshes.

$E_{11}(p) = E_1(p) + L_1 i_1(0) + E_3(p)$ = the algebraic sum of the LT of emfs of the first mesh.

$E_{22}(p) = E_2(p) - \dfrac{V_{C_2}(0)}{p} + E_3(p) + L_3 i_3(0)$ = the algebraic sum of the LT of emfs of the second mesh.

The solution of the set (3.27) is

$$I_1(p) = \frac{\Delta_1(p)}{\Delta(p)};\ I_2(p) = \frac{\Delta_2(p)}{\Delta(p)}, \tag{3.28}$$

where

$$\Delta(p) = \begin{vmatrix} Z_{11}(p) & Z_{12}(p) \\ Z_{21}(p) & Z_{22}(p) \end{vmatrix} = Z_{11}(p)Z_{22}(p) - Z_{12}(p)Z_{12}(p);$$

$$\Delta_1(p) = \begin{vmatrix} E_{11}(p) & Z_{12}(p) \\ E_{22}(p) & Z_{22}(p) \end{vmatrix} = E_{11}(p)Z_{22}(p) - E_{22}(p)Z_{21}(p);$$

$$\Delta_2(p) = \begin{vmatrix} Z_{11}(p) & E_{11}(p) \\ Z_{21}(p) & E_{22}(p) \end{vmatrix} = E_{22}(p)Z_{11}(p) - E_{11}(p)Z_{21}(p).$$

In order to transform currents $I_1(p),\ I_2(p)$ into the time solution it is possible to use (3.2) or the partial fraction expansion method that to be discussed in the next chapter.

### 3.7. The Partial Fraction Expansion Method

Due to write the time function that corresponds to the operational solution it is necessary to represent the operational solution as the sum of rational functions for which the time functions are known.

Suppose the operational solution is a proper rational function of the form

$$F(p) = \frac{Q(p)}{H(p)} = \frac{b_m p^m + b_{m-1} p^{m-i} + b_{m-2} p^{m-2} + \cdots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + a_{n-2} p^{n-2} + \cdots + a_1 p + a_0} \tag{3.29}$$

in which m<n and the polynomial

$$H(p) = a_n p^n + a_{n-1} p^{n-1} + a_{n-2} p^{n-2} + \cdots + a_1 p + a_0 = 0 \tag{3.30}$$

has the roots $p_1, p_2, \cdots, p_n$. Then the rational function 3.29 may be expanded into partial fractions

$$\frac{Q(p)}{H(p)} = \frac{A_1}{p - p_1} + \frac{A_1}{p - p_2} + \cdots + \frac{A_n}{p - p_n} = \sum_{k=1}^{n} \frac{A_k}{p - p_k}\ , \tag{3.31}$$

where

$$\frac{A_k}{p - p_k} = \frac{Q(p_k)}{H'(p_k)} \cdot \frac{1}{p - p_k}.$$

Because of

$$\frac{A_k}{p - p_k} \Leftrightarrow A_k e^{p_k t}$$

the searching time function will be

$$f(t) = \sum_{k=1}^{n} \frac{Q(p_k)}{H'(p_k)} e^{p_k t}. \tag{3.32}$$

This formula expresses the Heaviside expansion theorem.

The expansion (3.31) has as many terms as there are roots of the equation (3.30).

If one of roots is p=0, then corresponding term on the right – hand side of eq. (3.32) will be the component of the transient current or voltage due to a DC applied voltage. In this case the time function may be written as

$$f(t) = \frac{Q(0)}{H'(0)} + \sum_{k=2}^{n} \frac{Q(p_k)}{H'(p_k)} e^{p_k t} \tag{3.32}$$

If the eq. (3.30) also has complex conjugate roots, the corresponding terms in (3.32) will be also complex conjugates and their sum will be a real term.

When the drive is sinusoidal function then the eq.(3.30) has the imaginary conjugate roots $p_1 = j\omega;\ p_2 = -j\omega$. The time function is

$$f(t) = \frac{Q(j\omega)}{H'(j\omega)} e^{j\omega t} + \frac{Q(-j\omega)}{H'(-j\omega)} e^{-j\omega t} + \sum_{k=3}^{n} \frac{Q(p)}{H'(p)} e^{p_k t} \qquad 3.33$$

The first and second term of the right – hand side of eq. (3.33) express the sinusoidal component of the time function.

Now let's consider an examples.

In the network of Fig. 3.7 the current of current source increases linearly with time $i(t) = 2.5t;\ R = 40\Omega,\ C = 2\mu F$. Find the time dependence for the current $i_1$ through resistance R.

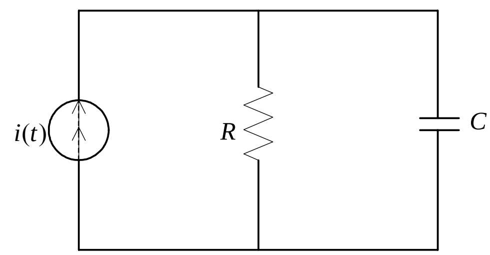



Fig.3.7

Solution: a transform of the current i(t) is

$$I(p) = \frac{2.5}{p^2}$$

The LT impedance of R and C connected in parallel is

$$Z(p) = \frac{R\frac{1}{Cp}}{R + \frac{1}{Cp}} = \frac{R}{RCp + 1}$$

The current transform of $i_1(t)$ is

$$I_1(p) = \frac{I(p)Z(p)}{R} = \frac{2.5}{RC} \cdot \frac{1}{p^2(p+a)}$$

$$a = \frac{1}{RC} = 12.55 \times 10^3 s^{-1} \qquad \frac{2.5}{RC} = 31.3 \times 10^3 s^{-1}$$

Then

$$\frac{1}{p^2(p+a))} \Leftrightarrow \frac{t}{a} - \frac{1}{a^2}\left(1 - e^{-\alpha t}\right);\ \frac{t}{a} = 0.08t * 10^{-3};\ \frac{1}{a^2} = 0.00645 * 10^{-6}$$

The transient current is

$$i_1(t) = 31.3\left[0.08t - 0.00645 * 10^{-3}\left(1 - e^{-12.05*10^{3} t}\right)\right].$$

## 4.Transients in the Case of Impulse Drives

### 4.1. The Impulse EMF′s and Systems

Sometimes a network is driven by emf′s the acting time of which is the value that to be the same order as the transient period of the network. This kind of emf′s and current and voltages caused by them are called as the impulse emf′s, currents and voltages and the networks with impulse emf' s, currents and voltages – as the impulse networks or impulse systems.

The impulse elektric values are characterized by the form, the peak value, the duration of pulse, the duration of the time interval between pulses. The form of some impulse functions as a rectangular (a), triangle (b), trapezoidal (c) and pulse train are shown in Fig. 4.1.

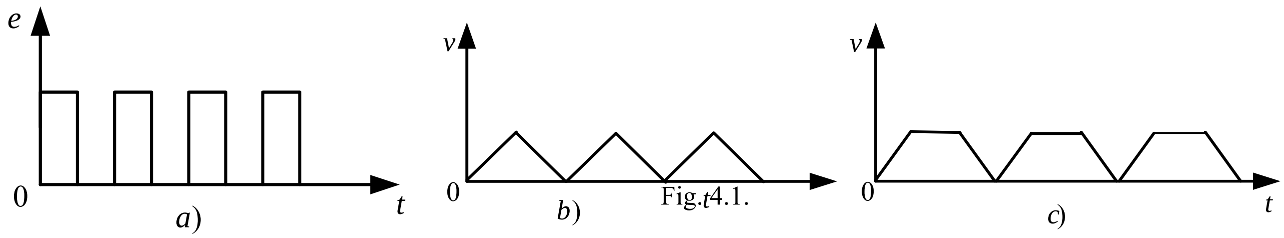



Fig.4.1.

### 4.2. The Pulse Functions

Several pulse functions are used due to analyse the impulse networks. They are ramp function, step function, delta function and so on.

The wave form of step function is shown in Fig. 11 It is called also as the impulse function of zero order. It may be written as

$$S(t) = \begin{cases} 0, t < 0 \\ a, t \geq 0. \end{cases} \qquad 4.1$$


Fig. 4.2

If a=1 it becomes to the unit step function and may be written as

$$1(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases} \qquad 4.2$$

The graph of a ramp function is shown in Fig.4.2 lt may be written as

$$R(t) = \begin{cases} 0, t < 0 \\ kt, t \geq 0 \end{cases} \qquad 4.3$$

There is the following relationship between the ramp and the step functions

$$\frac{d}{dt} R(t) = S(t); \quad \int_{-\infty}^{t} S(t) = R(t) \qquad 4.4$$

The waveform of a delta function is shown in Fig. 4.2. It may be written as

Fig.4.2

$$\delta(t)=\begin{cases}0, & t<0,\ t>0\\ \infty, & t=0\end{cases}. \qquad 4.5$$

The relationship between the unit step and delta functions are

$$\frac{dl(t)}{dt}=\delta(t); \qquad \int_{-\infty}^{t}\delta(t)=l(t) \qquad 4.6$$

In the case of integration of the unit delta function in the time interval $-\infty, +\infty$ we have

$$\int_{-\infty}^{+\infty}\delta(t)=1. \qquad 4.7$$

**4.3.The Transient Responses of the Network**

In the case of DC source applied to the input terminals of the circuit by a switch "s" as it shown in Fig. 4.3a.

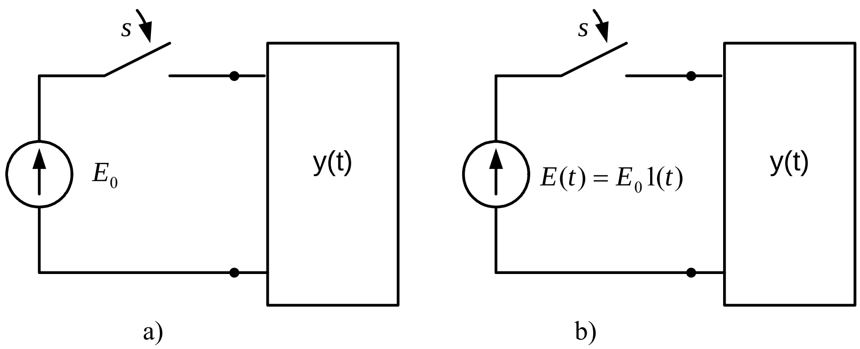


Fig.4.3

The driving emf may be represented by a step function (Fig.4.3b)

$$E(t)=E_0 1(t) \qquad 4.8$$

Transient current may be written as

$$i(t)=E(t)y(t)=\begin{cases}0, \ t<0\\ E_0 y(t) \quad t\geq 0\end{cases} \qquad 4.9$$

The time function y(t) is called as the unit step response and is defined as the ration of transient current to driving DC voltage

$$y(t)=\frac{i(t)}{E_0} \qquad 11\text{-}10$$

In general the unit step response of the network is defined as the ratio of the transient current that to be caused by DC voltage to the value of the DC voltage.

In the case of R-L circuit the transient current caused by the DC voltage $V_0$ is

$$i(t)=\frac{V_0}{R}\left(1-e^{-\frac{R}{L}t}\right) \qquad 4.12$$

and therefore the unit step response is

$$y(t)=\frac{i(t)}{V_0}=\frac{1}{R}\left(1-e^{-\frac{R}{L}t}\right) \qquad 4.13$$

In the case of R-C

$$y(t)=\frac{1}{R}e^{-\frac{R}{L}t} \qquad 4.14$$

In the case of R-C circuit

$$y(t)=\frac{1}{L(\alpha_1-\alpha_2)}\left(e^{\alpha_1 t}-e^{\alpha_2 t}\right) \qquad 4.15$$

where $\alpha_1$ and $\alpha_2$ are the roots of characteristic eguation.

If, instead of current, we are interested in voltage as the response of the network, we speak of the voltage transient response of that network.

Let a linear network under zero initial conditions be connected to a DC voltage source $V_0$. Then a voltage across a circuit element $V_a(t)$ is a function of time and is proportional to the $V_0$, or

$$V_a(t)=V_0k(t) \qquad 4.16$$

where k(t) is the voltage transfer ratio. It is a dimensionless quantity, numerically equal to the voltage across the “a” circuit element of the network when a Dc voltage of 1 volt is applied to its input.

Example. In the case of network shown in Fig. 4.4 determine the driving point tranient

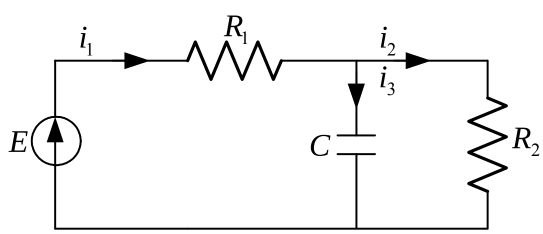



Fig. 4.4

response (the unit step response) y(t) and the voltage transfer ration $k_{vc}(t)$, if $R_1$=1.0 (k$\Omega$), $R_2$=1.0 (k$\Omega$) and C=50($\mu F$).

Solition: Using Kirchhoff's law we have

$$\left.\begin{aligned} &i_1-i_2-i_3=0\\ &E=R_1i_1+V_c \Rightarrow i_1=\frac{E-V_c}{R_1};\\ &R_2i_2-V_c=0 \quad \Rightarrow i_2=\frac{V_c}{R_2}; \end{aligned}\right\}$$

$$\frac{E-V_c}{R_1}-\frac{V_c}{R_2}-C\frac{dV_c}{dt}=0$$

$$C\frac{dV_C}{dt}+\frac{R_1+R_2}{R_1R_2}V_C=\frac{E}{R_1};\qquad \frac{R_1R_2}{R_1+R_2}=R_{12};$$

$$R_{12}C\frac{dV_C}{dt}+V_C=\frac{R_2}{R_1+R_2}E;$$

$$R_{12}C_\alpha+1=0\ \Rightarrow\ \alpha=-\frac{1}{R_{12}C};$$

$$V_C(t)=V_C+Ae^{\alpha t};$$

$$V_C'=\frac{R_2}{R_1+R_2}E;\quad V_C(t)=\frac{R_2}{R_1+R_2}E+Ae^{\alpha t}\Rightarrow 0=\frac{R_2}{R_1+R_2}E+A;$$

$$A=-\frac{R_2}{R_1+R_2}E;$$

$$V_C(t)=E\frac{R_2}{R_1+R_2}\left(1-e^{-\frac{t}{R_{12}C}}\right);$$

$$i_2(t)=\frac{V_C}{R_2}=E\frac{R_2}{R_1+R_2}\left(1-e^{-\frac{t}{R_{12}C}}\right);$$

$$i_3(t)=C\frac{dV_C}{dt}=CE\frac{R_2}{R_1+R_2}\cdot\frac{R_1+R_2}{R_1R_2C}e^{-\frac{t}{R_{12}C}}=\frac{E}{R_1}e^{-\frac{t}{R_{12}C}};$$

$$i_1(t)=i_2+i_3=\frac{E}{R_1+R_2}\left(1-e^{-\frac{t}{R_{12}C}}\right)+\frac{E}{R_1}e^{-\frac{t}{R_{12}C}}=\frac{E}{R_1+R_2}\left(1-\frac{R_2}{R_1}e^{-\frac{t}{R_{12}C}}\right)$$

Putting E=1 volt in the above equations we obtain

$$y(t)=\frac{i_1(t)}{E}=\frac{1}{R_1+R_2}\left(1-e^{-\frac{t}{R_{12}C}}\right)=0.5*10^{-3}\left(1-e^{-40t}\right)(Sm);$$

$$k_{vc}(t)=\frac{V_C(t)}{E}=\frac{R_2}{R_1+R_2}\left(1-e^{-\frac{t}{R_{12}C}}\right)=0.5\left(1-e^{-40t}\right).$$

### 4.3. The Duhamel integral Method

A third method of transient analysis applicable to linear electric circuits is the Duhamelintegralmethod.In this method the variable with respect to which integration is performed is denoted “x”,while “t” is retained for the instant of time at which the current in a circuit is to be found.

Let a voltage v(x) is applied at time t=0 to a circuit under zero initial conditions (Fig. 4.7).

To find the current in the circuit at the time t, we approximate the input wave – form by series of steps and sum the currents due to the initial voltage v(0) and all other voltage steps occuring each with a time lag.

The voltage v(0) gives rise to a current v(0)y(0) in the circuit, where y(t) is the unit step response. At the time $x+\Delta x$ there is a voltage jump

$$\Delta v = \left(\frac{dv}{dx}\right)\Delta x = v'(x)\Delta x \qquad 4.17$$

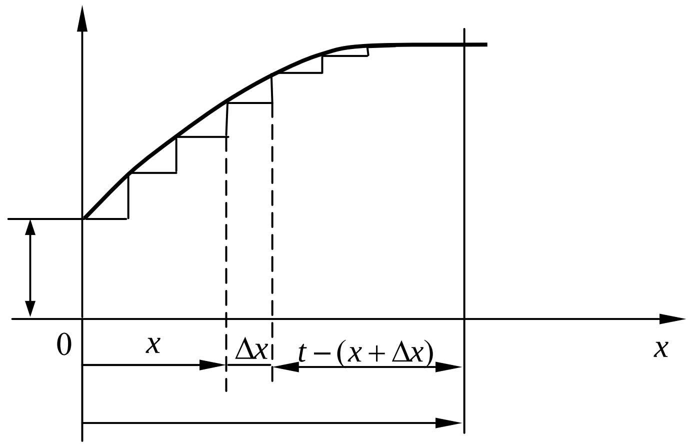



Fig.4.7

To find the current component due to this voltage jump $\Delta v$ at the time "t", we multiply the value $v'(x)\Delta x$ by the step response and take into account the time during which this jump is acting till the time "t"

$$v'(x)y(t - x - \Delta x) \qquad 4.18$$

The total current at the time "t" can be obtained by summing up the partial currents due to the individual voltage steps and by adding them to the current v(0)y(t)

$$i(t) = v(0)y(t) + \sum v'(x)y(t - x - \Delta x)\Delta x \qquad 4.19$$

The sum will have as many terms as there are voltage steps. It is obvious that the accuracy with which a series of steps can approximate a continuous input waveform increases with increasing number of steps. Accordingly, let up replace the finite time interval $\Delta x$ by an infinitly small one dx. Then the sum becomes an integral

$$i(t) = v(0)y(t) + \int_0^t v'(x)y(t - x)\Delta x \qquad 4.20$$

This formula is known as the Duhamel integral.

The stages in solution by the Duhamel integral method are follows:

1. The step response y(t) is found for the given network.
2. The function y(t-x) is found, for which purpose (t-x) is substitute by t in the expression for y(t).
3. The function $v^{/}(x)$ is found, for which purpose the derivative of the applied voltage v(t) with respect to time is found and in the relation thus obtained ‛t' is replaced dy ‛x'.
4. The function is steps (1), (2) and (3) above are substituted in aq. (11.20), the equation in integrated with respect to x and the limits are substituted.

Example: An applied voltage of R-C circuit is $v(t)=v_0(1-e^{-kt})$. Find the transient current.

Solution:

$$v(0) = 0;\ \ v'(t) = v_0 k e^{-kt};\ \ v'(x) = v_0 k e^{-kx}$$

$$y(t) = \frac{1}{R}e^{-\frac{t}{\tau}};\quad y(t - x) = \frac{1}{R}e^{-\frac{t-x}{\tau}}$$

$$i(t) = v(0)y(t) + \int_0^t v'(x)y(t - x)dx = \int v_0 k e^{-kx}\frac{1}{R}e^{-\frac{t-x}{\tau}}dx = \frac{v_0}{R}\frac{\tau}{\frac{1}{k} - \tau}\left(e^{-kt} - e^{-\frac{t}{\tau}}\right).$$

## 5. Two- Port Networks
### 5.1. Two Port Networks and Network Equations

A two-port network is an electric circuit with four terminals or two ports. Practical examples of to port networks are transformers, power transmission lines, bridge circuits and etc. They are shown in Fig.5.1

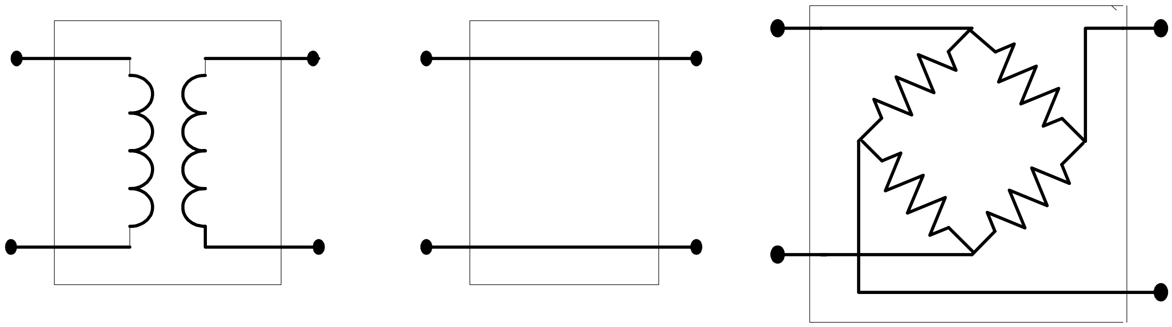


Fig.5.1

It is customary to symbolize a four terminal network by a box with two pairs of terminals , each pair making up a port.

If a two-port network contains a voltage or current source it is called an active two-port network, if not - a passive two-port network.

The terminals to which the energy source is connected are called the input terminals. Very often a four terminal network is an intermediate link between a source of supply and a load. The terminals to which the load is connected are referred to as the output terminals. Therefore current and voltage of the input are termed the input current and input voltage. Current and voltage of the output terminals are called the output current and output voltage

Let's derive two- port network equations.

Suppose a four terminal network is a link between a source of supply and a load. Then it forms with the energy source and the load a network with "m" number of loops. The corresponding set of network equations is .

$$\left.\begin{aligned}
&\dot{I}_1\underline{Z}_{11}+\dot{I}_2\underline{Z}_{12}+\dot{I}_3\underline{Z}_{13}+.............+\dot{I}_n\underline{Z}_{1m}=\dot{V}_1\\
&\dot{I}_1\underline{Z}_{21}+\dot{I}_2\underline{Z}_{22}+\dot{I}_3\underline{Z}_{23}+.............+\dot{I}_n\underline{Z}_{2m}=-\dot{V}_2\\
&\dot{I}_1\underline{Z}_{31}+\dot{I}_2\underline{Z}_{32}+\dot{I}_3\underline{Z}_{33}+.............+\dot{I}_n\underline{Z}_{3m}=0\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\dot{I}_1\underline{Z}_{m1}+\dot{I}_2\underline{Z}_{m2}+\dot{I}_3\underline{Z}_{m3}+.............+\dot{I}_n\underline{Z}_{mm}=0,
\end{aligned}\right\}\quad 5.1$$

where $Z_{11}$, $Z_{22}$, …..$Z_{nn}$ , $Z_{12,}$ $Z_{13}$ , ….$Z_{1m}$ , $Z_{m1}$, … are the self and mutual impedances of loops and $V_2 = I_2Z_L$ .

The solution of the above set of equations for loop currents of input and output circuits are:

$$\left.\begin{aligned}
\dot{I}_1&=\frac{\underline{\Delta}_{11}}{\underline{\Delta}}\dot{V}_1-\frac{\underline{\Delta}_{12}}{\underline{\Delta}}\dot{V}_2\\
\dot{I}_2&=\frac{\underline{\Delta}_{21}}{\underline{\Delta}}\dot{V}_1-\frac{\underline{\Delta}_{22}}{\underline{\Delta}}\dot{V}_2,
\end{aligned}\right\}\quad 5.2$$

where $\Delta$ is the determinant of the set (5.1) and $\Delta_{11}$, $\Delta_{12}$, $\Delta_{21}$, $\Delta_{22}$ are cofactors. The cofactors have dimension of conductance. Let's note them by $Y_{11}$, $Y_{12}$, $Y_{21}$ and $Y_{22}$ .Then the equations (5.2) may be written as

$$\left.\begin{aligned}
\dot{I}_1&=\underline{Y}_{11}\dot{V}_1+\underline{Y}_{12}\dot{V}_2\\
\dot{I}_2&=\underline{Y}_{21}\dot{V}_1+\underline{Y}_{22}\dot{V}_2.
\end{aligned}\right\}\quad 5.3$$

The above set is named the two-port network equations through why parameters. It allows to fined the input and output currents when the input and output voltages are given.

No let's solve the set (5.3) connection with voltages $V_1$ and $V_2$

$$\left.\begin{aligned} \dot{V}_1 &= \dot{I}_1 \frac{\underline{Y}_{22}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} - \dot{I}_2 \frac{\underline{Y}_{12}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} \\ \dot{V}_2 &= -\dot{I}_1 \frac{\underline{Y}_{21}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} + \dot{I}_2 \frac{\underline{Y}_{11}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} \end{aligned}\right\} \qquad 5.4$$

Using notations

$$\left.\begin{aligned} \frac{\underline{Y}_{22}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} &= \underline{Z}_{11}; \quad -\frac{\underline{Y}_{12}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} = \underline{Z}_{12}; \\ -\frac{\underline{Y}_{21}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} &= \underline{Z}_{21}; \quad \frac{\underline{Y}_{11}}{\underline{Y}_{11}\underline{Y}_{22}-\underline{Y}_{12}\underline{Y}_{21}} = \underline{Z}_{22} \end{aligned}\right\} \qquad 5.5$$

the set of equations (5.4) may be written as

$$\left.\begin{aligned} \dot{V}_1 &= \dot{I}_1\underline{Z}_{11} + \dot{I}_2\underline{Z}_{12} \\ \dot{V}_2 &= \dot{I}_1\underline{Z}_{21} + \dot{I}_2\underline{Z}_{22}. \end{aligned}\right\} \qquad 5.6$$

The above set is called the two-port network equations through "z" parameters.

If the set (5.6) solve connection with the input current and the input voltage then we'll have

$$\dot{I}_1 = \frac{1}{\underline{Z}_{21}}\dot{V}_2 - \frac{\underline{Z}_{22}}{\underline{Z}_{21}}\dot{I}_2. \qquad 5.7$$

Putting (5.7) in the first equation of the set (5.6) we have:

$$\dot{V}_1 = \frac{\underline{Z}_{11}}{\underline{Z}_{21}}\dot{V}_2 - \frac{\underline{Z}_{22}\underline{Z}_{11}}{\underline{Z}_{21}}\dot{I}_2 + \dot{I}_2\underline{Z}_{12} = \underline{A}\dot{V}_2 + \underline{B}\dot{I}_2 \qquad 5.8$$

The equations (5.7) and (5.8) may be written as

$$\left.\begin{aligned} \dot{V}_1 &= \underline{A}\dot{V}_2 + \underline{B}\dot{I}_2 \\ \dot{I}_1 &= \underline{C}\dot{V}_2 + \underline{D}\dot{I}_2, \end{aligned}\right\} \qquad 5.9$$

where $$\underline{A} = \frac{\underline{Z}_{11}}{\underline{Z}_{21}}; \quad \underline{B} = \underline{Z}_{12} - \frac{\underline{Z}_{11}\underline{Z}_{22}}{\underline{Z}_{21}}; \quad \underline{C} = \frac{1}{\underline{Z}_{21}}; \quad \underline{D} = -\frac{\underline{Z}_{22}}{\underline{Z}_{21}}. \qquad 5.10$$

Note that $\underline{A}\underline{D} - \underline{B}\underline{C} = 1$ 5.11

The equations (5.9) are called the "A" set of two- port network equations.

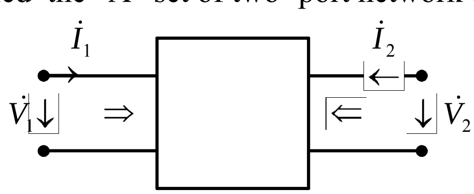


Fig.5.2

In the case of two source of supply when the energy is directed from the terminals of the ports (Fig.5.2) it is necessary to change the direction of current $I_2$ in all two-port network equations. Due to keep the two port network equations unchangeable it is sufficient to change the sign before parameters $Y_{21}$, $Y_{22}$, $Z_{12}$, $Z_{22}$, B and D. Then $Y_{12}=Y_{21}$, $Z_{12}=Z_{21}$ and

$$\underline{A}\underline{D} - \underline{B}\underline{C} = -1$$

Example 5.1. Find the "Z" parameters of the two port network of Fig.5.3

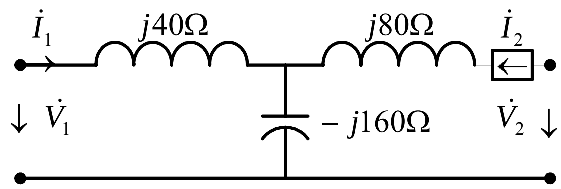



Fig.5.3

Solution:

From the two- port network equations (12.6)

$$\underline{Z}_{11} = \frac{\dot V_1}{\dot I_1} \quad at\, I_2 = 0; \qquad \underline{Z}_{22} = \frac{\dot V_2}{\dot I_2} \quad at\ \ I_1 = 0;$$

$$\underline{Z}_{12} = \frac{\dot V_1}{\dot I_2} \quad at\ \ I_1 = 0; \qquad \underline{Z}_{21} = \frac{\dot V_2}{\dot I_1} \quad at\ \ I_2 = 0.$$

Therefore due to find $Z_{11}$ and $Z_{21}$ we keep output open circuited

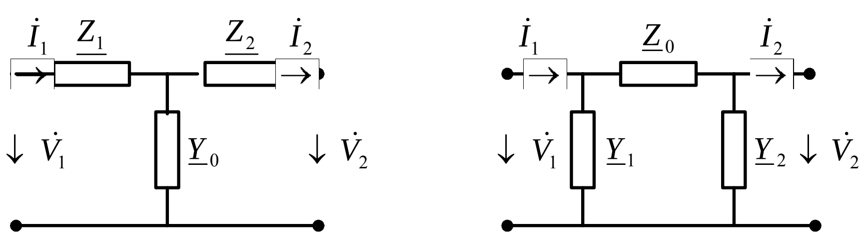
$$\underline{Z}_{11} = \frac{\dot V_1}{\dot V_1/(j40 - j!160)} = -j120\Omega; \qquad \underline{Z}_{21} = \frac{\dot I_1(-j160)}{\dot I_1} = -j160\Omega.$$

For calculation of $Z_{22}$ and $Z_{12}$ we shall have to keep the input open circuited and a source $V_2$ will have to be connected at output port 2

$$\underline{Z}_{22} = \frac{\dot V_2}{\dot V_2/(j80 - j160)} = -j80\Omega; \qquad \underline{Z}_{12} = \frac{\dot I_2(-j160)}{\dot I_2} = -j160\Omega.$$

## 5.2. Equivalent Circuits of a Passive Two- Port Networks

Any passive network can be replaced by an equivalent “T” or “П” networks shown in Fig. 5.4 a and b.




a) b)

Fig.5.4

In the equivalent “T” or “П” network the three parameters should be chosen so that the equivalent network would have the same general circuit parameters, for example A, B, C and D.

Let's start by the “T” network. According to the Kirchhoff's laws the input voltage and input current (Fig.5.4a) may be written as

$$\left.\begin{aligned} \dot V_1 &= \dot I_1 \underline{Z}_1 + \dot I_2 \underline{Z}_2 + \dot V_2 \\ \dot I_1 &= \dot I_2 + (\dot V_2 + \dot I_2 \underline{Z}_2)\underline{Y}_0 = \dot V_2 \underline{Y}_0 + \dot I_2 (1 + \underline{Z}_2 \underline{Y}_0). \end{aligned}\right\} \qquad 5.12$$

Putting the expression of the input current in the first equation of the above set , we have:

$$\dot V_1 = \dot V_2\left(1 + \underline{Z}_1 \underline{Y}_0\right) + \dot I_2\left(\underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1 \underline{Z}_2 \underline{Y}_0\right). \qquad 5.13$$

Comparing the equation (5.13) and the second equation of the set (5.12) with the set (5.11) we'll find out that

$$\underline{A} = 1 + \underline{Z}_1 \underline{Y}_0; \quad \underline{B} = \underline{Z}_1 + \underline{Z}_2 + \underline{Z}_1 \underline{Z}_2 \underline{Y}_0; \quad \underline{C} = \underline{Y}_0; \quad \underline{D} = 1 + \underline{Z}_2 \underline{Y}_0. \qquad 5.14$$

In the case of symmetrical network $Z_1 = Z_2$ and therefore A = D.

From the equations (5.14) it is possible to determine "T" equivalent network parameters when ABCD (transmission) parameters are given

$$\underline{Y}_0 = \underline{C}; \qquad \underline{Z}_1 \frac{\underline{A}-1}{\underline{C}}; \qquad \underline{Z}_2 = \frac{\underline{D}-1}{\underline{C}}; \qquad 5.15$$

Now let's conceder "Π" two-port network and express the input voltage and current by the output voltage and current using Kirchhoff's laws. According to the circuit of Fig.5.4b we have:

$$\left.\begin{aligned} \dot{V}_1 &= \underline{Z}_0\left(\dot{I}_2 + \dot{V}_2\underline{Y}_2\right) + \dot{V}_2 = \dot{V}_2\left(1 + \underline{Z}_0\underline{Y}_2\right) + \dot{I}_2\underline{Z}_0 \\ \dot{I}_1 &= \dot{V}_1\underline{Y}_1 + \dot{I}_1 + \dot{V}_2\underline{Y}_2 = \dot{V}_2\left(\underline{Y}_1 + \underline{Y}_2 + \underline{Z}_0\underline{Y}_1\underline{Y}_2\right) + \dot{I}_2\underline{Z}_0\underline{Y}_1. \end{aligned}\right\} \qquad 5.16$$

Again comparison of the equations (5.11) and (5.16) shows that the relationships between ABCD parameters and the parameters of "Π" equivalent two-port network are:

$$\underline{A} = \underline{Z}_0\underline{Y}_2; \qquad \underline{B} = \underline{Z}_0; \qquad \underline{C} = \underline{Y}_1 + \underline{Y}_2 + \underline{Z}_0\underline{Y}_1\underline{Y}_2; \qquad \underline{D} = \underline{Z}_0\underline{Y}_1 \qquad 5.17$$

When "Π" two port network is symmetrical $Y_1 = Y_2$ and therefore A=D.

From the equations (5.17) it is possible to find "Π" equivalent two-port network parameters when ABCD parameters are given

$$\underline{Z}_0 = \underline{B}; \qquad \underline{Y}_1 = \frac{\underline{D}}{\underline{B}}; \qquad \underline{Y}_2 \frac{\underline{A}}{\underline{B}}. \qquad 5.18$$

### 5.3. Determination of Transmission Parameters

The ABCD parameters may be found either by calculation if the internal connections and the circuit parameters are known or by the test when the driving-point impedances are defined measuring current, voltage and power or the phase displacement.

Experimentally the driving- point impedances are determined with the aid of an ammeter, a voltmeter and a wattmeter using the test of open circuit and short circuit.

The circuit diagram of the open circuit test is shown in Fig.5.5

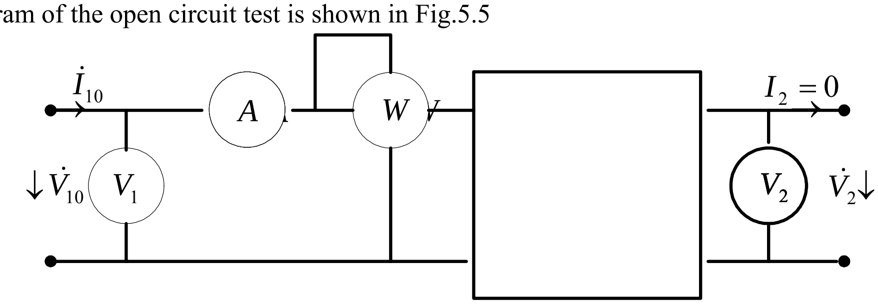



Fig.5.5

The value of input voltage $V_{10}$ must be chosen so that the output voltage will be equal to the normal voltage rating of the given two-port network. The values $V_{10}$ and $I_{10}$ are called the open circuit input voltage and current.. Using of readings of instruments the driving point open circuit impedance may be found as

$$\underline{Z}_{10} = \frac{V_{10}}{I_{10}} e^{j\varphi_{10}}, \qquad 5.19$$

where $\varphi_{10} = \cos^{-1}\frac{P_{10}}{V_{10}I_{10}}.$

Using the set of "A" parameter two-port network equations (5.11) in the case of the open circuit test we have

$$\dot{V}_{10} = \underline{A}\dot{V}_2$$

$$\dot{I}_{10} = \underline{C}\dot{V}_2,$$

from which

$$\frac{\underline{A}}{\underline{C}} = \frac{V_{10}}{I_{10}} e^{j\varphi_{10}}. \qquad 5.20$$

The curcuit diagram of the short circuit test is shown in Fig. 5.6.

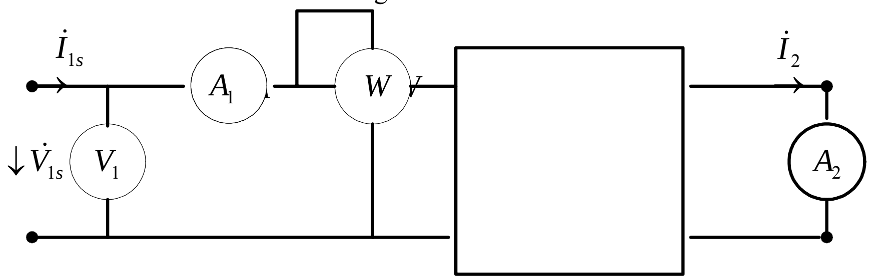



Fig.5.6

The driving voltage $V_{1s}$ is chosen so that the output current $I_2$ will be the normal current rating of the given two-port network. The values $V_{1s}$ and $I_{1s}$ are called the short circuit input voltage and current. Using the readings of instruments the driving point short circuit impedance may be found as

$$\underline{Z}_{1s} = \frac{V_{1s}}{I_{1s}} e^{j\varphi_{1s}}, \qquad 5.21$$

where $\varphi_{1s} = \cos^{-1}\frac{P_{1s}}{V_{1s}I_{1s}}$.

According to the set of equations (5.11) in the case of the short circuit test we have

$$\dot{V}_{1s} = \underline{B}\dot{I}_2$$

$$\dot{I}_{1s} = \underline{D}\dot{I}_2,$$

from which

$$\frac{\underline{B}}{\underline{D}} = \frac{V_{1s}}{I_{1s}} e^{j\varphi_{1s}}. \qquad 5.22$$

Using (5.20) and (5.22) for the symmetrical two-port network may be written the following set of equations

$$\left.\begin{aligned} &\frac{\underline{A}}{\underline{C}} = \frac{V_{10}}{I_{10}} e^{j\varphi_{10}} \\ &\frac{\underline{B}}{\underline{D}} = \frac{V_{1s}}{I_{1s}} e^{j\varphi_{1s}} \\ &\underline{A}\underline{D} - \underline{B}\underline{C} = 1 \\ &\underline{A} = \underline{D}. \end{aligned}\right\} \qquad 5.23$$

This set may be used for determination of "A" parameters.

In the case of unsymmetrical two-port network (A≠D) it is necessary to make supplementary test – open circuit or short circuit test when the input and output terminals of the network are interchanged. The results of this test allows to write the equation instead of the fourth equation of the set (5.23).

Example: The readings of instruments at open circuit test of the symmetrical two-port network are of an ammeter – 0.1A, a voltmeter – 200V and phasometer – ($-75^o$). Corresponding readings of instruments at short circuit test are: $I_{1s}$=0.80A, $V_{1s}$=16V. $\varphi_{1s}$=85$^o$. Calculate the transmission parameters.

Solution:

According to the set of equations (5.23) the following equations may be written:

$$\frac{\underline{A}}{\underline{C}} = \frac{200}{0.1} e^{-j75^0}\,\Omega; \quad \frac{\underline{B}}{\underline{D}} = \frac{16}{0.80} e^{j85^0}\,\Omega; \quad \underline{A}^2 - \underline{B}\underline{C} = 1,$$

solution of which gives:

$$\underline{A} = 0.995\angle 0.965^0; \quad \underline{B} = 19.9\angle 86^0\,\Omega; \quad \underline{C} = 0.497\angle 76^0\, mSm.$$

## 5.4. Differentiating and Integrating Circuits

In practice it is often desired to find the derivative and integral of a current or voltage acting in electric circuit. An elementary differentiator is shown in Fig.5.7. It is a two-port network that contains a coil and a resistor (Fig.5.7a) or a capacitor and a resistor (Fig.5.7b).

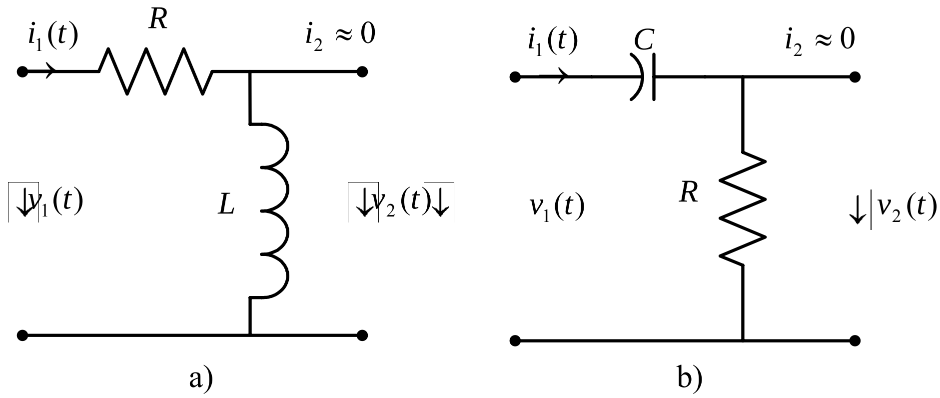



Fig.5.7

Let's show that the circuit of Fig.5.7a really is the differentiating circuit. According Kirchhoff's voltage law we have

$$L\frac{di}{dt} + Ri = v_1(t). \qquad 5.24$$

If the circuit parameters are chosen so that $v_L$ is much smaller than Ri, then approximately

$$Ri \approx v_1(t) \quad and \; i = \frac{v_1(t)}{R}.$$

Then the output voltage of two-port network is

$$v_2(t) = v_L = L\frac{di}{dt} \approx \frac{L}{R}\frac{dv_1}{dt} \qquad 5.25$$

and the output voltage is proportional to the derivative of the input voltage. The factor L/R=τ is the time constant of the differentiating circuit.

Differentiation may also be done by the circuit of Fig. 5.7b. The network equation is

$$v_1(t) = v_C + Ri_1 \qquad 5.26$$

and the output voltage is

$$v_2(t) = Ri_1.$$

If the circuit parameters are chosen so that $v_R$ is much smaller than $v_C$ than approximately

$$v_1(t) \approx v_C$$

and the output voltage is

$$v_2(t) = Ri_1 = RC\frac{dv_C}{dt} \approx \tau\frac{dv_1}{dt}, \qquad 5.27$$

where τ=RC is the time constant. Therefore the output voltage is proportional to the derivative of the input voltage.

Due to provide good quality of differentiation the time constant of the differentiating circuit must be very small. It must be the time interval duration of which the increment or decrement of the input voltage would be negligible compared with its peak value. This requirement may be written also as

$$\frac{k\omega L}{R} << 1; \qquad k\omega CR << 1, \qquad 5.28$$

where kω is the highs harmonic component of the input voltage.

Fig.5.8 shows the two-port networks, whose output voltages will under certain conditions be proportional to the integral of the input voltage.

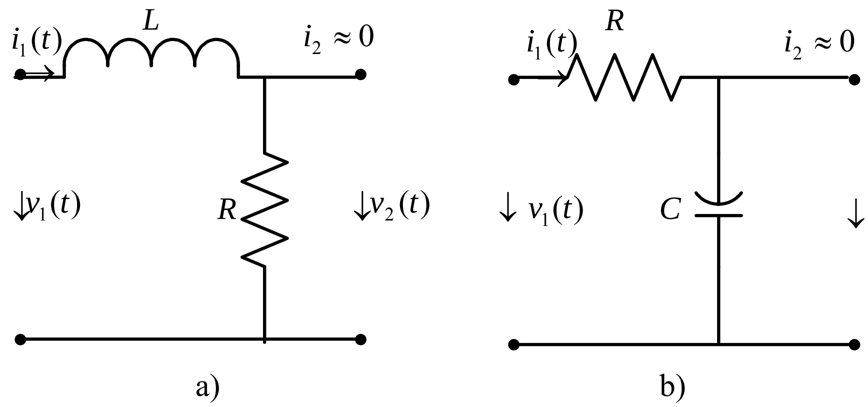



Fig.5.8

For the circuit of Fig.5.8a according to the Kirchhoff's voltage law we have

$$v_1(t) = L\frac{di_1}{dt} + Ri_1. \qquad 5.29$$

When the circuit parameters are chosen so that $v_L >> v_R$ then

$$v_1(t) \approx v_L L\frac{di_1}{dt}$$

and the output voltage is

$$v_2(t) = Ri_1 \approx \frac{R}{L}\int v_1(t)dt. \qquad 5.30$$

Therefore the output voltage is proportional to the integral of the input voltage.

In the case of Fig.12.8b the network equation is

$$v_1(t) = Ri_1 + \frac{1}{C}\int i_1 dt. \qquad 5.31$$

If the circuit parameters are chosen so that $v_R >> v_C$ then

$$v_1(t) \approx v_R = Ri_1 \qquad \Rightarrow \quad i_1 \approx \frac{v_1(t)}{R}.$$

Then the output voltage is

$$v_2(t) = v_C = \frac{1}{C}\int i_1 dt \approx \frac{1}{RC}\int v_1(t)dt. \qquad 5.32$$

It means that the output voltage is proportional to the integral of the input voltage. Therefore the tow-port network to be concidered also may be used as an integrating circuit. Again due to get good quality of integration the time constant of the integrator must be very large. It must be the time interval duration of which the input variable value would make at least one full cycle of variation.

## 5.5. Characteristic Impedance and Transmission Constant

The ratio of the input and output voltage of a two-port network is termed the driving point impedance

$$\underline{Z}_1 = \frac{\dot{V}_1}{\dot{I}_1} = \frac{\underline{A}\dot{V}_2 + \underline{B}\dot{I}_2}{\underline{C}\dot{V}_2 + \underline{D}\dot{I}_2}. \qquad 5.33$$

Since $\dot{V}_2 = \dot{I}_2 \underline{Z}_L$ then the driving point impedance of the two-port network will depend on the load impedance $Z_L$

$$\underline{Z}_1 = \frac{\underline{A}\underline{Z}_L + \underline{B}}{\underline{C}\underline{Z}_L + \underline{D}} = \underline{Z}_{C1}. \qquad 5.34$$

When $\underline{Z}_L = \underline{Z}_{C1}$ it is called the matching load. Let $\dot{V}_2 / \dot{I}_2 = \underline{Z}_{C2}$. Then the eq. (5.33) ma be written as

$$\underline{Z}_{C1} = \frac{\underline{A}\underline{Z}_{C2} + \underline{B}}{\underline{C}\underline{Z}_{C2} + \underline{D}}. \qquad 5.35$$

If interchange the input and output terminals the "A" parameter network equations will be

$$\left.\begin{aligned} \dot{V}_2 &= \underline{D}\dot{V}_2 + \underline{B}\dot{I}_2 \\ \dot{I}_2 &= \underline{C}\dot{V}_2 + \underline{A}\dot{I}_2, \end{aligned}\right\}$$

Then the characteristic impedance from the side of output terminals is

$$\underline{Z}_{C2} = \frac{\underline{D}\underline{Z}_{C1} + \underline{B}}{\underline{C}\underline{Z}_{C1} + \underline{A}}. \qquad 5.36$$

Together solution of (12.35) and (12.36) gives the characteristic impedances

$$\underline{Z}_{C1} = \sqrt{\frac{\underline{A}\underline{B}}{\underline{C}\underline{D}}}; \qquad \underline{Z}_{C2} = \sqrt{\frac{\underline{D}\underline{B}}{\underline{C}\underline{A}}}. \qquad 5.37$$

If a two-port network is a symmetrical one then $\underline{Z}_{C1} = \underline{Z}_{C2} = \underline{Z}_C = \sqrt{\underline{B}/\underline{C}}$. This value is termed the repeated impedance because if loading the two-port network by $Z_C$ the driving point impedance of the two-port network is $Z_c$.

When a symmetrical two-port network is loaded by $Z_C$ then

$$\dot{V}_1 = \underline{A}\dot{V}_2 + \underline{B}\dot{I}_2 = \underline{A}\dot{V}_2 + \underline{B}\frac{\dot{V}_2}{\underline{Z}_L} = \dot{V}_2\left(\underline{A} + \sqrt{\underline{B}\underline{C}}\right)$$

$$\dot{I}_1 = \underline{C}\dot{V}_2 + \underline{D}\dot{I}_2 = \underline{C}\dot{I}_2\underline{Z}_C + \underline{D}\dot{I}_2 = \dot{I}_2\left(\underline{A} + \sqrt{\underline{B}\underline{C}}\right).$$

From above equations follows that

$$\frac{\dot{V}_1}{\dot{V}_2} = \frac{\dot{I}_1}{\dot{I}_2} = \underline{A} + \sqrt{\underline{B}\underline{C}}. \qquad 5.38$$

The complex value

$$\underline{A} + \sqrt{\underline{B}\underline{C}} = e^{\underline{g}},$$

where $\underline{g} = \ln(\underline{A} + \sqrt{\underline{B}\underline{C}}) = a + jb$ is named the transmission constant.

From the equation (5.38) follows that

$$\dot{V}_1 = e^a e^{jb}\dot{V}_2 \quad \text{and} \quad \dot{I}_1 = e^a e^{jb}\dot{I}_2.$$

It means that the magnitude of $V_1$ is $e^a$ times the magnitude of $V_2$. Then $V_2$ lags $V_1$ by the angle "b". The ratio

$$\frac{V_1}{V_2} = \frac{I}{I_2} = e^a \quad \Rightarrow \quad a = \ln\frac{V_1}{V_2} = \ln\frac{I_1}{I_2}.$$

The value "a" is the damping factor. Its unit is nepper. When the damping factor is one nepper then the amplitude of the output voltage is 2.712 times less than the amplitude of the input voltage.

The nepper is very large unit and in practice the decibel is used instead. The decibel is defined from the equality: $a = 20\lg V_1/V_2$. If $a = 1$ then $V_1/V_2 = 10^{1/20} \approx 1.12$. It means that when a two-port network has a damping factor 1dec., then the output voltage is by 12% less then the input voltage. The values of $Z_C$ , "g", "a" and "b" are called the secondary parameters of a two-port network.

Let's expressed the transmission parameters by the hyperbolic functions and the secondary parameters of a two-port network. For the symmetrical two-port network we have

$$\underline{A}^2 - \underline{B}\underline{C} = 1 \qquad or \qquad (\underline{A} + \sqrt{\underline{B}\underline{C}})(\underline{A} - \sqrt{\underline{B}\underline{C}}) = 1$$

Then

$$\underline{A} - \sqrt{\underline{B}\underline{C}} = \frac{1}{\underline{A} + \sqrt{\underline{B}\underline{C}}} = e^{-\underline{g}}; \qquad \underline{A} + \sqrt{\underline{B}\underline{C}} = e^{\underline{g}}.$$

But according to the definition of hyperbolic functions

$$sh\underline{g} = \frac{e^{\underline{g}} - e^{-\underline{g}}}{2} = \sqrt{\underline{B}\underline{C}}; \qquad ch\underline{g} = \frac{e^{\underline{g}} + e^{-\underline{g}}}{2} = \underline{A}. \tag{5.39}$$

The taking into account that $\underline{Z}_c = \sqrt{\underline{B}/\underline{C}}$ the "B" and "C" parameters are

$$\underline{B} = \underline{Z}_C sh\underline{g}; \qquad \underline{C} = \frac{sh\underline{g}}{\underline{Z}_C}.$$

The two-port network equations through "A" parameters will be

$$\left.\begin{aligned} \dot{V}_1 &= \dot{V}_2 ch\underline{g} + \dot{I}_2 \underline{Z}_C sh\underline{g} \\ \dot{I}_1 &= \dot{V}_2 \frac{sh\underline{g}}{\underline{Z}_C} + \dot{I}_2 ch\underline{g} \end{aligned}\right\} \tag{5.40}$$

The driving point impedance is the result of division the first equation by the second one in the above set and then division the right hand side of the obtained equality by the value $\dot{I}_2 ch\underline{g}$

$$\underline{Z}_1 = \frac{\dot{V}_1}{\dot{I}_1} = \underline{Z}_C \frac{\underline{Z}_L + \underline{Z}_C th\underline{g}}{\underline{Z}_L th\underline{g} + \underline{Z}_C}. \tag{5.41}$$

At open circuit test when $Z_L = \infty$ the open circuit driving point impedance is

$$\underline{Z}_{10} = \frac{\underline{Z}_C}{th\underline{g}} \tag{5.42}$$

and at short circuit test when $Z_L = 0$ the short circuit driving point impedance is

$$\underline{Z}_{1s} = \underline{Z}_C th\underline{g} \tag{5.43}$$

The results of the open and short circuit tests may be used due to find the secondary parameters of a two-port network

$$\underline{Z}_C = \sqrt{\underline{Z}_{10}\underline{Z}_{1s}}; \qquad th\underline{g} = \sqrt{\frac{\underline{Z}_{1s}}{\underline{Z}_{10}}} \tag{5.44}$$

It is also possible to find the driving point impedance if the open and short circuit impedances are known. Putting (12.42) and (12.43) in (12.41) we have

$$\underline{Z}_1 = \underline{Z}_{10} \frac{\underline{Z}_L + \underline{Z}_{1s}}{\underline{Z}_L + \underline{Z}_{10}}. \tag{5.45}$$

## 5.6. Connection of Two-port Networks

For studies a complicated network it is usually divided in to several two-port networks

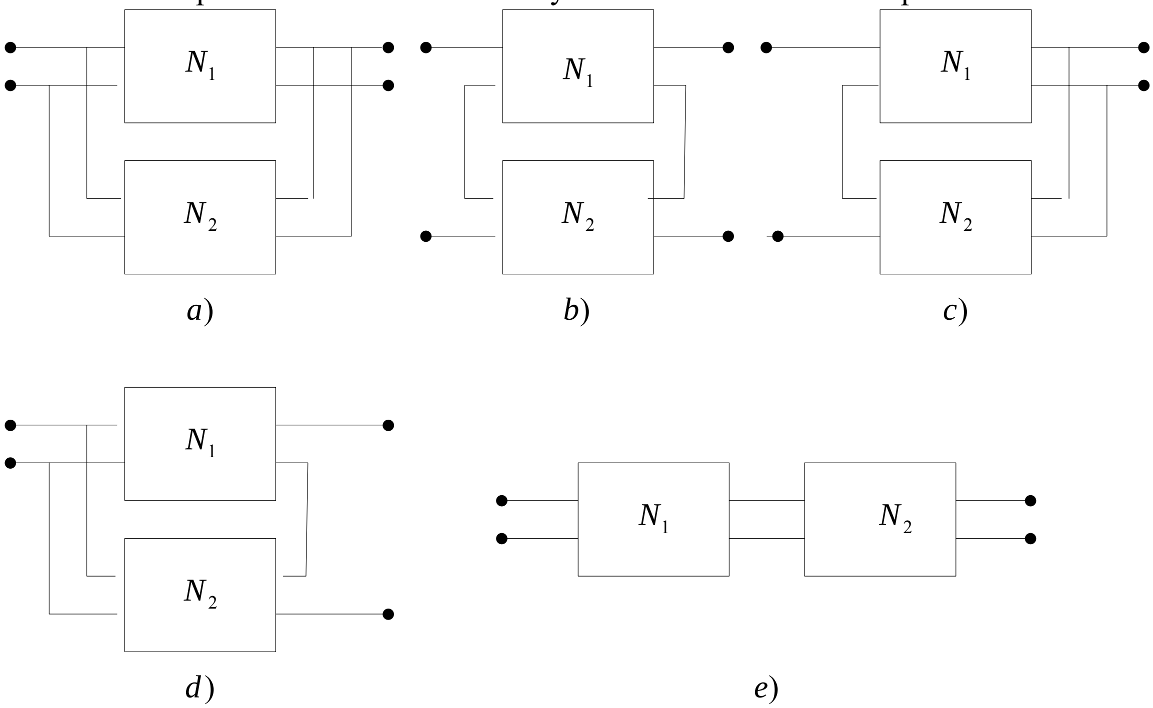



Fig.5.9

to be connected in parallel (a), series (b), series-parallel (c), parallel-series (d) or cascade (e). It is customary to use at parallel connection "Y" two-port network equations, at series connection – "Z" equations, at series-parallel connection – "H" equations, at parallel-series connection – "G" equations and at cascade connection – "A" equations.

Let's consider the method of replacement of two parallel or cascade connected two-port networks by the equivalent one. Fig.5.10 shows parallel connected networks $N_a$ and $N_b$ , network equations of which written through. "Y" parameters are

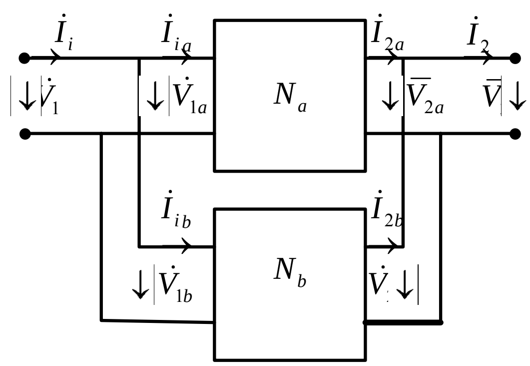



Fig.5.10

$$\dot{I}_{1a} = \underline{Y}_{11a}\dot{V}_{1a} + \underline{Y}_{12a}\dot{V}_{2a} \qquad \dot{I}_{1b} = \underline{Y}_{11b}\dot{V}_{1b} + \underline{Y}_{12b}\dot{V}_{2b}$$

$$\dot{I}_{2a} = \underline{Y}_{21a}\dot{V}_{1a} + \underline{Y}_{22a}\dot{V}_{2a} \qquad \dot{I}_{2b} = \underline{Y}_{21b}\dot{V}_{1b} + \underline{Y}_{22b}\dot{V}_{2b}$$

Then because of $\dot{I}_1 = \dot{I}_{1a} + \dot{I}_{1b}$ *and* $\dot{I}_2 = \dot{I}_{2a} + \dot{I}_{2b}$ in matrix form we have

$$\begin{bmatrix}\dot{I}\\ \dot{I}\end{bmatrix}=\begin{bmatrix}\dot{I}_{1a}\\ \dot{I}_{2a}\end{bmatrix}+\begin{bmatrix}\dot{I}_{1b}\\ \dot{I}_{2b}\end{bmatrix}=\begin{bmatrix}\underline{Y}_{11a} & \underline{Y}_{12a}\\ \underline{Y}_{21a} & \underline{Y}_{22a}\end{bmatrix}\begin{bmatrix}\dot{V}_{1a}\\ \dot{V}_{2a}\end{bmatrix}+\begin{bmatrix}\underline{Y}_{11b} & \underline{Y}_{12b}\\ \underline{Y}_{21b} & \underline{Y}_{22b}\end{bmatrix}\begin{bmatrix}\dot{V}_{1b}\\ \dot{V}_{2b}\end{bmatrix}=$$

$$=\left\{\begin{bmatrix}\underline{Y}_{11a} & \underline{Y}_{12a}\\ \underline{Y}_{21a} & \underline{Y}_{22a}\end{bmatrix}+\begin{bmatrix}\underline{Y}_{11b} & \underline{Y}_{12b}\\ \underline{Y}_{21b} & \underline{Y}_{22b}\end{bmatrix}\right\}\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\end{bmatrix}=\begin{bmatrix}\underline{Y}_{11a}+\underline{Y}_{11b} & \underline{Y}_{12a}+\underline{Y}_{12b}\\ \underline{Y}_{21a}+\underline{Y}_{21b} & \underline{Y}_{22a}+\underline{Y}_{22b}\end{bmatrix}\begin{bmatrix}\dot{V}_1\\ \dot{V}_2\end{bmatrix}.$$

So the matrix of the equivalent two-port network is the sum of matrixes of two-port networks connected in parallel.

For the cascade connected networks (Fig.5.11) due to replace them by the equivalent network let's write their matrix equations through transmission parameters:

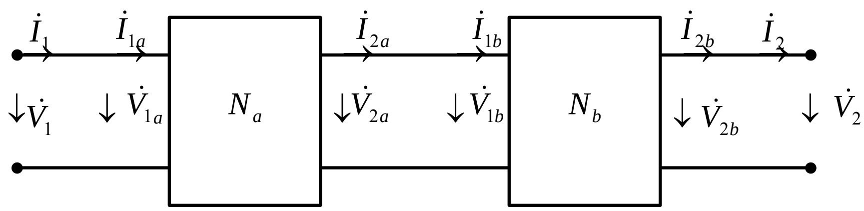



Fig.5.11

$$\begin{bmatrix}\dot{V}_1\\ \dot{I}_1\end{bmatrix}=\begin{bmatrix}\dot{V}_{1a}\\ \dot{I}_{1a}\end{bmatrix}=\begin{bmatrix}\underline{A}_a & \underline{B}_a\\ \underline{C}_a & \underline{D}_a\end{bmatrix}\begin{bmatrix}\dot{V}_{2a}\\ \dot{I}_{2a}\end{bmatrix}=\begin{bmatrix}\underline{A}_a & \underline{B}_a\\ \underline{C}_a & \underline{D}_a\end{bmatrix}\begin{bmatrix}\dot{V}_{1b}\\ \dot{I}_{1b}\end{bmatrix}=\begin{bmatrix}\underline{A}_a & \underline{B}_a\\ \underline{C}_a & \underline{D}_a\end{bmatrix}\begin{bmatrix}\underline{A}_b & \underline{B}_b\\ \underline{C}_b & \underline{D}_b\end{bmatrix}\begin{bmatrix}\dot{V}_{2b}\\ \dot{I}_{2b}\end{bmatrix}=$$

$$=\begin{bmatrix}\underline{A} & \underline{B}\\ \underline{C} & \underline{D}\end{bmatrix}\begin{bmatrix}\dot{V}_2\\ \dot{I}_2\end{bmatrix},$$

where

$$\begin{bmatrix}\underline{A} & \underline{B}\\ \underline{C} & \underline{D}\end{bmatrix}=\begin{bmatrix}\underline{A}_a & \underline{B}_a\\ \underline{C}_a & \underline{D}_a\end{bmatrix}\begin{bmatrix}\underline{A}_b & \underline{B}_b\\ \underline{C}_b & \underline{D}_b\end{bmatrix}=\begin{bmatrix}\underline{A}_a\underline{A}_b+\underline{B}_a\underline{C}_b & \underline{A}_a\underline{B}_b+\underline{B}_a\underline{D}_b\\ \underline{C}_a\underline{A}_b+\underline{D}_a\underline{C}_b & \underline{C}_a\underline{B}_b+\underline{D}_a\underline{D}_b\end{bmatrix}$$

is the product of two matrixes.

Problems:

5.1.Find the "Z" parameters for the network of Fig.5.12

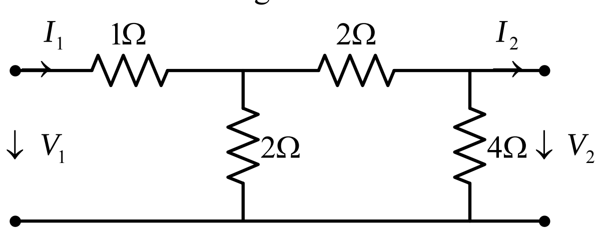



Fig.5.12

5.2.Find the "Y" parameters of the network of Fig.5.13

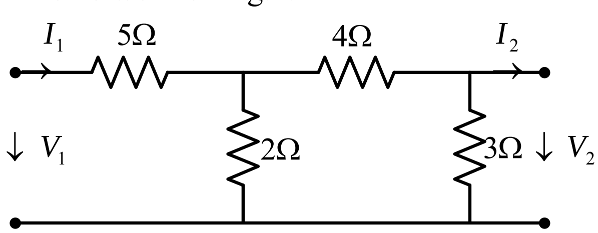



Fig.5.13

5.3.Calculate the open circuit and short circuit parameters of the network of Fig.5. 14

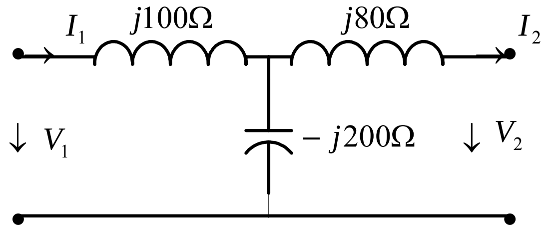


Fig.5.14

5.4. A symmetrical "T" section has the following open circuit and short circuit input impedances

$Z_{ino} = 800\Omega;$ $Z_{ins} = 600\Omega.$

Determine "T" section parameters.

# 6. Transmission Lines
## 6.1. Fundamental Differential Equations for Uniform Lines

The term of transmission line refers to a system of conductors used to transmit electric power or signal from a source to a load. The earlier transmission lines were developed to work at commercial frequencies 50 or 60Hz. Later on with the development of long distance communication systems had been designed transmission lines that to be worked at audio frequencies (20 to 20000Hz) and at radio frequencies 100kHz to 10GHz.

The study of transmission line is basically a study of the circuits with distributed constants- unlike the circuits with lumped parameters. As it's known the resistance, inductance and capacitance of the circuit with lumped parameters are concentrated or lumped at discrete points. In a transmission line they are distributed along the length of the line.

Basically all transmission lines are the same but they have different forms. A line with two parallel wires is a common form and is known as an open-wire line with air as dielectric. Ordinary telephone, telegraph and power transmission lines are of this form.

Another form of transmission line is a cable, which consists of hundreds of insulated conductors, twisted in pairs and covered with a protective lead tube. Such lines are used for telephone transmission. Cables are also used for power transmission; such cables are constructed with two, three or four conductors having large cross sectional area. A third form of transmission line is a coaxial line which consists of a hollow tube as one conductor, the second conductor being placed axially at the centre of the tube. Such lines are used for high radio frequency work.

A line must be considered as a circuit with distributed parameters if the length of the line is compared with the wave length ($\lambda$) of the waves to be propagated over it. If a line has the length more then at least $\lambda/10$ , this line must be considered as a transmission (long) line.

In the transmission line the current and voltage vary continuously along the conductor. If the coordinate axis 0 – x is directed along the line current and voltage are functions of to variables i(x,t), v(x, t).

Let $R_0$ be the series resistance per unit length of the line, $L_0$ – the inductance per unit length, $C_0$ – the capacitance per unit length and $G_0$ – shunt conductance per unit length. If parameters per unit length of the line are constant values along the line ( they don' t depend on the variable "x" such transmission line is referred to as uniform. Units of parameters are Ω/m, H/m, F/m Sm/m.

Let the line be divided into the sections of length "dx" (fig.13.1), where "x" is the distance from the origin of the set of coordinate.

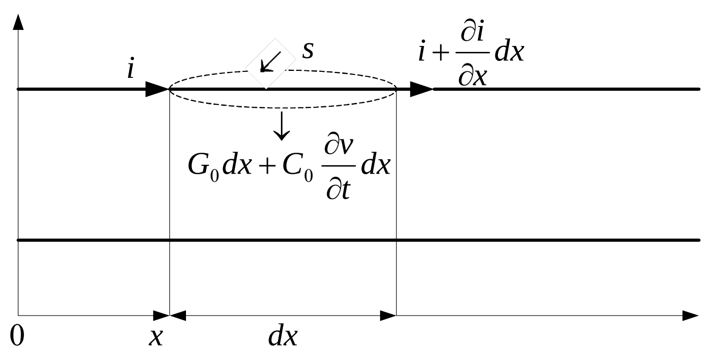



Fig.6.1

The resistance over dx is $R_0$dx; the inductance is $L_0$dx; the leakage conductance is $G_0$dx and the capacitance – $C_0$dx. The instantaneous current at the beginning of the section dx is i and at a distance dx further along the line the current will be

$$i+\frac{\partial i}{\partial x}dx.$$

due to leakage through the leakage conductance and capacitance between wires. The rate of change of current multiplied by the distance dx gives the increment current over the distance dx.

$$\frac{\partial i}{\partial x}dx = vG_0dx+\frac{\partial v}{\partial t}C_0dx.$$

By Kirchhoff's current law for the closed surface "s" we have

$$-i+\left(vG_0dx+\frac{\partial v}{\partial t}C_0dx\right)+\left(i+\frac{\partial i}{\partial x}dx\right)=0$$

and division this equation by dx gives

$$-\frac{\partial i}{\partial x} = G_0 v + C_0 \frac{\partial v}{\partial t}. \qquad 6.1$$

Similarly, the voltage drop across the section dx is (Fig.6.2)

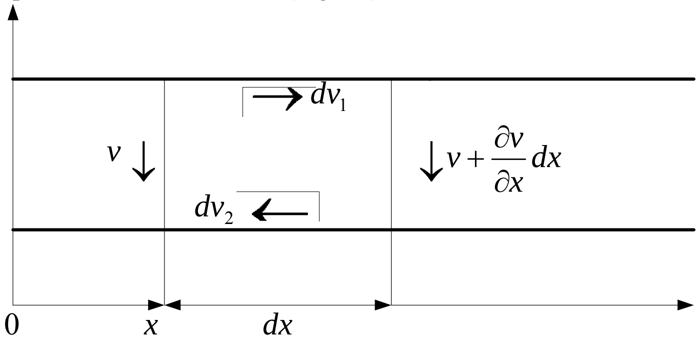



Fig. 6.2

$$dv_1 + dv_2 = iR_0 dx + L_0 dx \frac{\partial i}{\partial t}.$$

Application of Kirchhoff's voltage law to the closed loop formed by the section of the length dx gives

$$-v_1 + \left( iR_0 dx + L_0 dx \frac{\partial i}{\partial t} \right) + \left( v + \frac{\partial v}{\partial x} dx \right) = 0.$$

Simplification and division by dx give

$$-\frac{\partial v}{\partial x} = R_0 i + L_0 \frac{\partial i}{\partial t}. \qquad 6.2$$

Equations(6.1) and (6.2) are the fundamental differential equations of a line. They contain partial derivatives of "v" and "i " with respect to time "t" and distance "x". Therefore voltage and current are functions of two variables. The section dx may be represented by the following circuit diagram

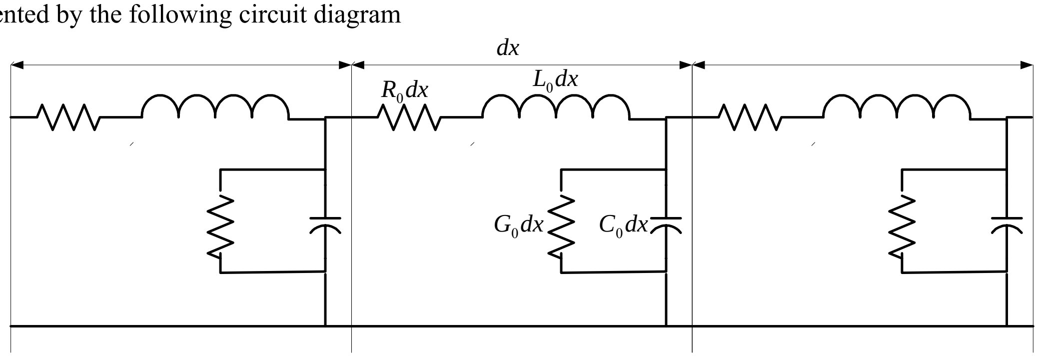



Fig.6.3

This circuit is called as an electrical model of a transmission line.

## 6.2. Sinusoidal Steady-State Performance of Transmission Line

Let the voltage and current in a line vary with time according to the sine law. Then they may be represented by the complex voltage and complex current

$$\left. \begin{aligned} \dot{I} e^{j\omega t} &\Leftrightarrow I_m \sin(\omega t + \psi_i), \\ \dot{V} e^{j\omega t} &\Leftrightarrow V_m \sin(\omega t + \psi_v), \end{aligned} \right\} \qquad 6.3$$

where

$$\dot{I} = \frac{\dot{I}_m}{\sqrt{2}} = \frac{I_m}{\sqrt{2}} e^{j\psi_i}; \qquad \dot{V} = \frac{V_m}{\sqrt{2}} e^{j\psi_v}.$$

Complex current and voltage are functions of the distance "x" , but not of time. The factor $e^{j\omega t}$ is a function of time, but it is independent on "x". With the above in mind, we may convert equations in partial derivatives into equations in simple derivatives

$$\left.\begin{array}{ll} \frac{\partial v}{\partial x} \Leftrightarrow e^{j\omega t}\frac{d\dot{V}}{dx}: & L_0\frac{\partial i}{\partial t} \Leftrightarrow L_0 j\omega \dot{I} e^{j\omega t} \\ \frac{\partial i}{\partial x} \Leftrightarrow e^{j\omega t}\frac{d\dot{I}}{dx}: & C_0\frac{\partial v}{\partial t} \Leftrightarrow C_0 j\omega \dot{I} e^{j\omega t}. \end{array}\right\} \qquad 6.4$$

Taking into account these relationships in equations (6.1) and (6.2) we have

$$\left.\begin{array}{l} -\frac{d\dot{I}}{dx} = \underline{Y}_0\dot{V}, \\ -\frac{d\dot{V}}{dx} = \underline{Z}_0\dot{I}, \end{array}\right\} \qquad 6.5$$

where

$$\left.\begin{array}{l} \underline{Y}_0 = G_0 + j\omega C_0, \\ \underline{Z}_0 = R_0 + j\omega L_0 \end{array}\right\} \qquad 6.6$$

are complex shunt admittance and complex series impedance accordingly

The values $G_0$, $C_0$, $L_0$, $R_0$, $Y_0$ and $Z_0$ are called as primary parameters of the transmission lines.

To solve the system of equations (6.5) for the complex voltage, we should. at first differentiate the second equation with respect to "x"

$$-\frac{d^2\dot{V}}{dx^2} = \underline{Z}_0\frac{d\dot{I}}{dx}. \qquad 6.7$$

Substitution the first equation of (6.5) in eq.(6.7) gives

$$\frac{d^2\dot{V}}{dx^2} - \underline{\gamma}\dot{V} = 0, \qquad 6.8$$

where the value

$$\underline{\gamma} = \sqrt{\underline{Y}_0\underline{Z}_0} = \sqrt{(G_0 + j\omega C_0)(R_0 + j\omega L_0)} \qquad 6.9$$

is called the propagation constant of a line. It is a complex value

$$\underline{\gamma} = \alpha + j\beta, \qquad 6.10$$

where $\alpha$ is the attenuation constant,

$\beta$ - the phase shift constant of the line.

Eq. (13.8) is a second-order linear differential equation the solution of which is

$$\dot{V} = \underline{A}_1 e^{-\underline{\gamma}x} + \underline{A}_2 e^{-\underline{\gamma}x}. \qquad 6.11$$

Here $A_1$ and $A_2$ are the integration constants.

The current can be found from the second eq. (6.5)

$$\dot{I} = -\frac{1}{\underline{Z}_0}\frac{d\dot{V}}{dx} = \frac{\underline{\gamma}}{\underline{Z}_0}\left(\underline{A}_1 e^{-\underline{\gamma}x} - \underline{A}_2 e^{-\underline{\gamma}x}\right) = \frac{1}{\underline{Z}_c}\left(\underline{A}_1 e^{-\underline{\gamma}x} - \underline{A}_2 e^{-\underline{\gamma}x}\right), \qquad 6.12$$

where

$$\underline{Z}_c = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} \qquad 6.13$$

is called the characteristic impedance. Its unit is ohm., while the values $\alpha$, $\beta$ and $\gamma$ have dimension of $m^{-1}$. The values $\alpha$, $\beta$, $\gamma$ and $Z_c$ are called the secondary parameters of the transmission line.

Let us find the integration constants. Suppose complex voltage and current are known at the sending end of the transmission line, where x=0. From the equations (6.11) and (6.12) it follows that

$$\left.\begin{array}{l} \dot{V}_1 = \underline{A}_1 + \underline{A}_2 \\ \dot{I}_1 = \frac{1}{\underline{Z}_c}\left(\underline{A}_1 + \underline{A}_2\right) \end{array}\right\} \qquad 6.14$$

Solution of this set of equations gives

$$\left.\begin{aligned} \underline{A}_1 &= \frac{1}{2}\left(\dot{V}_1 + \dot{I}_1 \underline{Z}_c\right) \\ \underline{A}_2 &= \frac{1}{2}\left(\dot{V}_1 - \dot{I}_1 \underline{Z}_c\right) \end{aligned}\right\} \qquad 6.15$$

Substitution of eq. (6.15) in eq. (6.11) and eq. (6.12) gives

$$\left.\begin{aligned} \dot{V} &= \frac{1}{2}\left(\dot{V}_1 + \dot{I}_1 \underline{Z}_c\right)e^{-\underline{\gamma} x} + \frac{1}{2}\left(\dot{V}_1 - \dot{I}_1 \underline{Z}_c\right)e^{\underline{\gamma} x} \\ \dot{I} &= \frac{1}{\underline{Z}_c}\left[\frac{1}{2}\left(\dot{V}_1 + \dot{I}_1 \underline{Z}_c\right)e^{-\underline{\gamma} x} - \frac{1}{2}\left(\dot{V}_1 - \dot{I}_1 \underline{Z}_c\right)e^{\underline{\gamma} x}\right]. \end{aligned}\right\} \qquad 6.16$$

These equations may be written as

$$\left.\begin{aligned} \dot{V} &= \dot{V}_1 \frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2} - \dot{I}_1 \underline{Z}_c \frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2} \\ \dot{I} &= -\frac{\dot{V}_1}{\underline{Z}_c} \frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2} + \dot{I}_1 \frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2}. \end{aligned}\right\} \qquad 6.17$$

Let's introduce the hyperbolic functions

$$ch\underline{\gamma} x = \frac{e^{\underline{\gamma} x} + e^{-\underline{\gamma} x}}{2}, \qquad sh\underline{\gamma} x = \frac{e^{\underline{\gamma} x} - e^{-\underline{\gamma} x}}{2}. \qquad 6.18$$

Then instead of eq.(6.17) we have

$$\left.\begin{aligned} \dot{V} &= \dot{V}_1 ch\underline{\gamma} x - \dot{I}_1 \underline{Z}_c sh\underline{\gamma} x \\ \dot{I} &= -\frac{\dot{V}_1}{\underline{Z}_c} sh\underline{\gamma} x + \dot{I}_1 ch\underline{\gamma} x. \end{aligned}\right\} \qquad 6.19$$

At the end of the line x=l voltage and current according to the equations (6.19) will be

$$\left.\begin{aligned} \dot{V}_2 &= \dot{V}_1 ch\underline{\gamma} l - \dot{I}_1 \underline{Z}_c sh\underline{\gamma} l \\ \dot{I}_2 &= -\frac{\dot{V}_1}{\underline{Z}_c} sh\underline{\gamma} l + \dot{I}_1 ch\underline{\gamma} l. \end{aligned}\right\} \qquad 6.20$$

From these equations it's possible to express the voltage and current at the beginning of the line by the voltage and current at the end of the line

$$\left.\begin{aligned} \dot{V}_1 &= \dot{V}_2 ch\underline{\gamma} l + \dot{I}_2 \underline{Z}_c sh\underline{\gamma} l \\ \dot{I}_1 &= \frac{\dot{V}_2}{\underline{Z}_c} sh\underline{\gamma} l + \dot{I}_2 ch\underline{\gamma} l. \end{aligned}\right\} \qquad 6.21$$

These equations look like two-port network equations with transmission parameters

$$\left.\begin{aligned} \dot{V}_1 &= \underline{A}\dot{V}_2 + \underline{B}\dot{I}_2 \\ \dot{I}_1 &= \underline{C}\dot{V}_2 + \underline{D}\dot{I}_2, \end{aligned}\right\} \qquad 6.22$$

where

$$\left.\begin{aligned} \underline{A} &= \underline{D} = ch\underline{\gamma} l, \\ \underline{B} &= \underline{Z}_c sh\underline{\gamma} l, \\ \underline{C} &= \frac{1}{\underline{Z}_c} sh\underline{\gamma} l \end{aligned}\right\} \qquad 6.23$$

It is clear, that transmission line is a symmetrical to-port network (A=D). Then the following equality takes place

$$\underline{AD} - \underline{BC} = ch^2 \underline{\gamma} l - sh^2 \underline{\gamma} l = 1.$$

Therefore the transmission line again may be considered as a two-port network so if it is necessary it may be represented by "T" or "π" equivalent.

**Example 6.1**. The transmission line having the length l=5km and the secondary parameters $\underline{Z}_c = 500\angle -37^0\Omega,\ \underline{\gamma} = 0.1414 + j0.144km^{-1}$ is loaded at the end of the line by resistance of $R_L$=400 $\Omega$. Find $U_1$ and $I_1$ if the load current $I_2 = 0.5\angle 0^0\,A$ and frequency f=1000Hz.

Solution:

$$\dot{U}_1 = \dot{U}_2 ch\underline{\gamma}l + \dot{I}_2\ \underline{Z}_c sh\underline{\gamma}l;$$

$$\underline{\gamma}i = (0.1414 + j0.1414)5 = 0.707 + j0.707;$$

$$ch\underline{\gamma}l = 0.5(e^{\underline{\gamma}l} + e^{-\underline{\gamma}l}) = 1.07\angle 27{,}3^0;\quad sh\underline{\gamma}l = 0.5(e^{\underline{\gamma}l} - e^{-\underline{\gamma}l}) = 1*\angle 54.3^0);$$

$$\dot{U}_1 = 200*1.07\angle 27.3^0 + 0.5\angle 0^0*500\angle -37^0*1\angle 54.3^0 = 463\angle 22^0(V);$$

$$\dot{I}_1 = \frac{\dot{U}_2}{\underline{Z}_c} sh\underline{\gamma}l + \dot{I}_2 ch\underline{\gamma}l = 0.8\angle 53.7^0(A).$$

## 6.3. Driving Point Impedance of a Line

Due to get the driving point impedance of o line it is enough to divide the first equation of the set (6.21) by the second one

$$\underline{Z}_{in} = \frac{\dot{V}_2 ch\underline{\gamma}l + \dot{I}_2\underline{Z}_c sh\underline{\gamma}l}{\dot{V}_2\dfrac{sh\underline{\gamma}l}{\underline{Z}_c} + \dot{I}_2 ch\underline{\gamma}l} = \underline{Z}_c\frac{\underline{Z}_L + \underline{Z}_c th\underline{\gamma}l}{\underline{Z}_L th\underline{\gamma}l + \underline{Z}_c}, \qquad 6.24$$

where the input impedance off a load

$$\underline{Z}_L = \frac{\dot{V}_2}{\dot{I}_2} \qquad 6.25$$

There are several possible way of termination of transmission lines:

1. The load impedance is infinite (open circuit);
2. The load impedance zero (short circuit);
3. The load impedance is matched (that is $Z_L = Z_c$). .

In the case of open circuit the output current $I_2 = 0$ and the driving point impedance

$$\underline{Z}_{ino} = \frac{\dot{V}_{10}}{\dot{I}_{10}} = \underline{Z}_c\frac{1}{th\underline{\gamma}l}. \qquad 6.26$$

In the case of short circuit the output voltage $V_2 = 0$ and the driving point impedance

$$\underline{Z}_{ins} = \frac{\dot{V}_{1s}}{\dot{I}_{1s}} = \underline{Z}_c th\underline{\gamma}l. \qquad 6.27$$

Here $V_{10}$, $I_{10}$ are the input open circuit voltage and current and $V_{1s}$ , $I_{1s}$ are the input short circuit voltage and current of the line.

Multiplication of the eq. (6.26) by the eq.(6.27) gives the formula of the characteristic impedance of the line

$$\underline{Z}_c = \sqrt{\underline{Z}_{ino}\underline{Z}_{ins}}. \qquad 6.28$$

Division of the eq. (6.27) by the eq. (6.26) gives the formula for calculating of thγl

$$th\underline{\gamma}l = \sqrt{\frac{\underline{Z}_{ins}}{\underline{Z}_{ino}}} \qquad 6.29$$

When $Z_L=Z_c$ (matched load) the driving point impedance from the eq. (6.24) $Z_{in}=Z_c$ and it is the repeating impedance.

Now let's get formula that allows to calculate the propagating constant, attenuation and phase shift factor when the value thγl is known.

$$th\underline{\gamma}l = \frac{e^{\underline{\gamma}l} - e^{-\underline{\gamma}l}}{e^{\underline{\gamma}l} + e^{-\underline{\gamma}l}} = \frac{1 - e^{-2\underline{\gamma}l}}{1 + e^{-2\underline{\gamma}l}} = Te^{j\tau}. \qquad 6.30$$

From this equality it is possible to write that

$$1 - e^{-2\underline{\gamma}l} = Te^{j\eta} + Te^{j\eta}e^{-2\underline{\gamma}l} \Rightarrow e^{2\underline{\gamma}l} = \frac{1 + Te^{j\tau}}{1 - Te^{j\tau}} = We^{j\xi}. \qquad 6.31$$

Then

$$\ln\frac{1+Te^{j\tau}}{1-Te^{j\tau}} = 2\underline{\gamma}l; \qquad \Rightarrow \qquad \underline{\gamma}l = \frac{1}{2}\ln\left(We^{j\tau}\right) = \frac{\ln W}{2} + j\frac{\xi}{2} = (\alpha + j\beta)l. \qquad 6.32$$

Therefore

$$\left.\begin{aligned} \alpha l &= \frac{\ln W}{2}; \qquad \Rightarrow \qquad \alpha = \frac{\ln W}{2} \\ \beta l &= \frac{\xi + 2\pi k}{2} \qquad \Rightarrow \qquad \beta = \frac{1}{l}(\pi k + \frac{\xi}{2}), \end{aligned}\right\} \qquad 6.33$$

where k=0, 1, 2, …

## 6.4. Incident and Reflected Waves

If the two wires of a long line, are suddenly connected to a source of supply, an energy wave advance along the line towards the receiving end at a certain conditions, such a wave may be reflected at the receiving end of the line, thereby giving rise to a reflected wave. Let's conceder the problem of incident and reflected waves. According to the formulas (6.16) and (6.17) the complex voltage of the line is the following sum

$$\dot{V} = \dot{V}_{\varphi} + \dot{V}_{\psi}, \qquad 6.34$$

where

$$\left.\begin{aligned} \dot{V}_{\varphi} &= \frac{1}{2}\left(\dot{V}_1 + \dot{I}_1\underline{Z}_c\right)e^{-\underline{\gamma}x} \\ \dot{V}_{\psi} &= \frac{1}{2}\left(\dot{V}_1 - \dot{I}_1\underline{Z}_c\right)e^{\underline{\varphi}x}. \end{aligned}\right\} \qquad 6.35$$

Suppose that

$$\left.\begin{aligned} \frac{1}{2}\left(\dot{V}_1 + \dot{I}_1\underline{Z}_c\right) &= \dot{V}_{\varphi 1} = V_{\varphi 1}e^{j\xi}, \\ \frac{1}{2}\left(\dot{V}_1 - \dot{I}_1\underline{Z}_c\right) &= \dot{V}_{\psi 1} = V_{\psi 1}e^{j\eta}. \end{aligned}\right\} \qquad 6.36$$

Taking into account that

$$\underline{\gamma} = \alpha + j\beta \qquad 6.37$$

the complex voltage may be written as

$$\dot{V} = \dot{V}_{\varphi} + \dot{V}_{\psi} = V_{\varphi 1}e^{j\xi}e^{-\alpha x}e^{-j\beta x} + V_{\psi 1}e^{j\eta}e^{\alpha x}e^{j\beta x} = V_{\varphi 1}e^{-\alpha x}e^{-j(\xi-\beta x)} + V_{\psi 1}e^{\alpha x}e^{j(\eta+\beta x)}. \qquad 6.38$$

Appling the same procedure to the eq.(6.17) and also substituting

$$\underline{Z}_c = Z_c e^{j\varphi_c}, \qquad 6.39$$

we have

$$\dot{I} = I_{\varphi 1}e^{-\alpha x}e^{j(\xi-\beta x-\varphi_c)} - I_{\psi 1}e^{\alpha x}e^{j(\eta+\beta x-\varphi_c)}, \qquad 6.40$$

where

$$I_{\varphi 1} = \frac{V_{\varphi 1}}{Z_c}; \qquad I_{\psi 1} = \frac{V_{\psi 1}}{Z_c}. \qquad 6.41$$

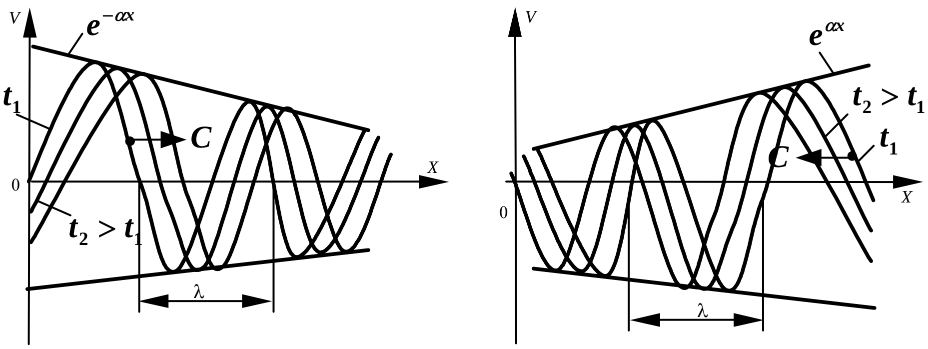
Corresponding of eq.(6.38) and (6.40) the time functions are

$$\left.\begin{aligned} v(x,t) &= \sqrt{2}V_{\varphi 1}e^{-\alpha x}\sin(\omega t + \xi - \beta x) + \sqrt{2}V_{\psi 1}\sin(\omega t + \eta + \beta x). \\ i(x,t) &= \sqrt{2}I_{\varphi 1}e^{-\alpha x}\sin(\omega t + \xi - \beta x - \varphi_c) - \sqrt{2}I_{\psi 1}\sin(\omega t + \eta + \beta x - \varphi_c). \end{aligned}\right\} \qquad 6.42$$

The first term of the first eq. (6.42) represents an incident voltage wave and the first term of the second eq. (6.42) represents an incident current wave. The second term of the first eq. (6.42) represents a reflected voltage wave and the second term of the second eq. (6.42) - a reflected current wave.

Each of the incident wave components is a sinusoidal wave the amplitude of which decreases as "x" increases according to the exponential law and the phase is a function of time and coordinate 'x".

Each of the reflected wave components decays as the wave travels from the receiving end to the sending end. Physically, the decrease in the amplitude of the incident and reflected waves as they advance along a line is due to losses in the line.




a) b)

Fig.6.4

Fig.6.4a shows the distribution of an incident voltage wave along the line as a function of “x”. The incident wave travels from the left to the right. The Fig.6.4b shows the distribution of a reflected voltage wave traveling from the right to the left.

The ratio of the reflected voltage wave to the incident voltage wave at the receiving end of a line is termed the voltage reflection coefficient

$$\underline{q}_v = \frac{\dot{V}_{\psi 2}}{\dot{V}_{\varphi 2}}. \qquad 6.43$$

Similarly, the current reflection coefficient is

$$\underline{q}_i = \frac{\dot{I}_{\psi 2}}{\dot{I}_{\varphi 2}}. \qquad 6.44$$

Complex voltage and current at the receiving end of a line are

$$\left.\begin{aligned} &\dot{V}_2 = \dot{V}_{\varphi 2} + \dot{V}_{\psi 2} \\ &\dot{I}_2 = \dot{I}_{\varphi 2} + \dot{I}_{\psi 2} = \frac{\dot{V}_{\varphi 2}}{\underline{Z}c} - \frac{\dot{V}_{\psi 2}}{\underline{Z}c}; \qquad \Rightarrow \ \dot{I}_2 \underline{Z}c = \dot{V}_{\varphi 2} - \dot{V}_{\psi 2}. \end{aligned}\right\} \qquad 6.45$$

Taking the sum and difference of equations of the set.(6.45) we'll have

$$\left.\begin{aligned} 2\dot{V}_{\varphi 2} &= \dot{V}_2 + \dot{I}_2 \underline{Z}_c = \dot{I}_2(\underline{Z}_L + \underline{Z}_c) \\ 2\dot{V}_{\psi 2} &= \dot{V}_2 - \dot{I}_2 \underline{Z}_c = \dot{I}_2(\underline{Z}_L - \underline{Z}_c). \end{aligned}\right\} \qquad 6.46$$

Therefore the voltage reflection coefficient may be defined as

$$\underline{q}_v = \frac{\underline{Z}_L - \underline{Z}_c}{\underline{Z}_L + \underline{Z}_c}. \qquad 6.47$$

It must be noted that the ratio of incident voltage wave to the incident current wave in any point of a line is the characteristic impedance

$$\frac{\dot{V}_\varphi}{\dot{I}_\varphi} = \underline{Z}_c \qquad 6.48$$

and the ratio of the reflected voltage wave to the reflected current wave is

$$\frac{\dot{V}_\psi}{\dot{I}_\psi} = -\underline{Z}_{c.}$$

Therefore the current reflection coefficient is

$$\underline{q}_i = \frac{\underline{Z}_c - \underline{Z}_L}{\underline{Z}_L + \underline{Z}_c} = -\underline{q}_v. \qquad 6.49$$

In the case of open circuit when $Z_L = \infty$, the voltage reflection coefficient $q_v = 1$, the current at the receiving end $I_2=0$ and the voltage

$$\dot{V}_2 = 2\dot{V}_{\varphi 2}. \qquad 6.50$$

In the case of short circuit $Z_L$ =0. the voltage reflection coefficient $q_v$=-1, the voltage at the receiving end $V_2$=0 and the current

$$\dot{I}_2 = 2\dot{I}_{\varphi 2}.$$

Usually a transmission line is a link between an energy source and a load. If the load impedance is other than the characteristic impedance of the line, the incident wave will be partly absorbed by the load and partly reflected from it, giving rise to a wave. To avoid reflections in the line it is customary to make $Z_L$=$Z_c$ . Such a load is termed matched and the procedure providing this condition is termed load matching. In this case the voltage reflection coefficient $q_v$=0 and there is no reflected voltage and current waves $V_\psi$=0, $I_\psi$=0.

## 6.5. Wave Length and Phase Velocity

The minimum distance between two consecutive points on a line at which a wave has the same phase is termed a wave length (Fig.13.4). The relationship between the wave length, period or frequency and velocity "c" is

$$\lambda = cT = \frac{c}{f}. \qquad 6.51$$

The phase velocity $c_{ph}$ may be defined as the speed with which an observer should travels along line to see the same phase of a wave. If the phase of an incident voltage wave is constant, then by the eq.(13.42) $\omega t + \xi - \beta x = const.$ The first derivative of this equality with respect to time is

$$\frac{d}{dt}(\omega t + \xi - \beta x) = \omega - \beta \frac{dx}{dt} = 0$$

or $$c_{ph} = \frac{dx}{dt} = \frac{\omega}{\beta}. \qquad 6.52$$

**Example 6.2**. Determine the wavelength of electromagnetic wave at frequencies f=50Hz and f=50MHz.

Solution: At f=50Hz $\lambda = \frac{3*10^8}{50} = 6000km.$

At f=50MHz $\lambda = \frac{3*10^8}{50*10^6} = 600m.$

## 6.6. All-Pass Lines

Due to transmit information by transmission (communication) line it is essential that an electromagnetic wave propagates along the line without any distortion. This quality makes a telephone line transmit the speech so that its frequency spectrum is the same both at the sending and the receiving ends of the line. The line like this is called as all-pass line.

For a transmission line to be an all-pass one it is essential that the attenuation constant and the phase velocity would be independent of the frequency. These requirements are satisfied when the line parameters are related thus:

$$\frac{R_0}{L_0} = \frac{G_0}{C_0} \qquad 6.53$$

In this case the characteristic impedance and the propagation constant are:

$$\left.\begin{aligned} \underline{Z}_C &= \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} = \sqrt{\frac{L_0}{C_0}}, \\ \underline{\gamma} &= \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)} = \sqrt{R_0 G_0} + j\omega\sqrt{L_0 C_0} = \alpha + j\beta. \end{aligned}\right\} \qquad 6.54$$

Therefore the characteristic impedance and attenuation don' t depend on frequency. As to phase velocity

$$c_{ph} = \frac{\omega}{\beta} = \frac{1}{\sqrt{L_0 C_0}} \qquad 6.55$$

again it does not depend on frequency. Hence if the condition (13.53) is satisfied and the line is matched then it is the all-pass line.

**Example 6.3**: The transmission line has primary parameters $R_0 = 3\Omega / km;\ L_0 = 2m\Omega / km;\ G_0 = 1\mu Sm / km;\ C_0 = 6nF / km.$ Find the supplementary inductance being switched on in the line after each kilometers that provides the line to be all pass line.

Solution: $L_s + L_0 = \dfrac{R_0 C_0}{G_0} = \dfrac{3*6*10^{-9}}{10^{-6}} = 18mH / km;\quad L_s = 18 - 2 = 16mH / km.$

## 6.7. The Lossless Line

Strictly speaking, lossless line doesn't exist in practice, but one can build a line in which losses are negligible ($R_0 << \omega L_0$ and $G_0 << \omega C_0$). In this case the characteristic impedance and the propagating constant are

$$\underline{Z}_C = \sqrt{\frac{R_0 + j\omega L_0}{G_0 + j\omega C_0}} \approx \sqrt{\frac{L_0}{C_0}}, \qquad \underline{\gamma} = \sqrt{(R_0 + j\omega L_0)(G_0 + j\omega C_0)} \neq j\omega\sqrt{L_0 C_0}. \qquad 6.56$$

Hence the attenuation and the characteristic impedance don't depend on frequency. As to the phase velocity

$$c_{ph} = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{L_0 C_0}}$$

it again doesn't depend on frequency and a lossless line may be consider as all-pass line.

As an example of lossless line let's take an open wire line (Fig.12.5a). An inductance and a capacitance of such line may be found as

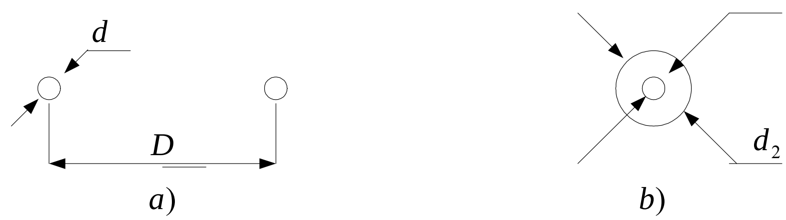



Fig.6.5

$$L_0 = \frac{\mu_0}{\pi}\ln\frac{2D}{d}, \qquad C_0 = \frac{\varepsilon_0 \pi}{\ln\frac{2D}{d}}, \qquad 6.57$$

where D = distance between wires,
d = diameter of the wire,
$\mu_0 = 4\pi*10^{-7}$ (H/m),
$\varepsilon_0 = 8.85*10^{-12}$(F/m).

The characteristic impedance of the line

$$Z_c = \sqrt{\frac{L_0}{C_0}} = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\ln\frac{2d}{d} = 120\ln\frac{2d}{d} \qquad 6.58$$

This value practically is in the range 300 – 400 Ω.

The phase velocity of the line

$$c_{ph} = \frac{1}{\sqrt{L_0 C_0}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3*10^8\, m/s. \qquad 6.59$$

As an another example of lossless line let's take a coaxial line of Fig.6.5b. It is supposed that $d_1$ is the diameter of the central conductor and $d_2$ – the diameter of the hollow tube (the second conductor). Inductance and capacitance of a coaxial cable are

$$L_0 = \frac{\mu\mu_0}{2\pi}\ln\frac{d_2}{d_1}, \qquad C_0 = \frac{2\pi\varepsilon\varepsilon_0}{\ln(d\ /d_1)}, \qquad 6.60$$

where μ = the relative permeability,

ε = the relative permittivity.

The characteristic impedance of the line is

$$Z_C\sqrt{\frac{L_0}{C_0}} = \frac{1}{2\pi}\sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}}\ln\frac{d_2}{d_1} = 60\sqrt{\frac{\mu}{\varepsilon}}\ln\frac{d_2}{d_1}. \qquad 6.61$$

This value is practically in the range 50 - 75 Ω.

The phase velocity of the line is

$$c_{ph} = \frac{1}{\sqrt{L_0C_0}} = \frac{1}{\sqrt{\mu\mu_0\varepsilon\varepsilon_0}} = \frac{3*10^8}{\sqrt{\mu\varepsilon}}. \qquad 6.62$$

Because of $\mu > 1$ and $\varepsilon > 1$ then $c_{ph} < 3*10^8$ m/s.

Example 6.4. Find the phase velocity for an open-air loss less transmission line having parameters: r=1.5mm; D=300mm.

Solution: The phase velocity

$$v_{ph} = \frac{\omega}{\beta} = \frac{1}{\sqrt{L_0C_0}};$$

$$L_0 = \frac{\mu_0}{\pi}\ln\frac{D}{r} = 2.12\mu H/m;$$

$$C_0 = \frac{\pi\varepsilon_0}{\ln\frac{D}{r}}5.24pF/m.$$

$$v_{ph} = \frac{1}{\sqrt{2.12*10^{-6}*5.24*10^{-12}}} = 3*10^8 m/s.$$

## 6.8. Standing Electromagnetic Waves

When a lossless line is an open or short –circuited at the receiving end there are reflected waves. If the incident wave and the reflected wave are of an equal magnitude (because of α=0), their combination produces standing wave. A standing voltage wave and a standing current wave combine a standing electromagnetic wave.

Mathematically standing voltage or current waves are described by the product of two periodical functions. One is a function of the coordinate of a point on the line, the other is a function of time. A standing voltage wave always lags or leads with the associated current wave be $90^o$.

The points on the line where the periodical function of a coordinate passes through zero are termed the nodes, and the points where the periodical function of a coordinate takes on a maximum value are called the loops or anti-nodes. So there are current and voltage nodes, current and voltage loops.

In the presence of standing waves, no electromagnetic energy is transmitted from the sending to the receiving end of the line. Instead, each quarter-wave section of the line stores up electromagnetic energy. This energy is exchanged periodically between the electric field and the magnetic field existing round the conductors of the line. When all of the energy goes into the electric field, the current along the line is zero and the voltage is a maximum. When all energy goes into the magnetic field the voltage across the line is zero and the current in the line takes on a maximum value.

Let's derive equations of standing wave in the cases of open circuited and short circuited.

The equations (6.11) and (6.12) form the following set

$$\left.\begin{aligned} \dot{V} &= \underline{A}_1 e^{-\underline{\gamma} x} + \underline{A}_2 e^{\underline{\gamma} x} \\ \dot{I} &= \frac{1}{\underline{Z}_C}\left(\underline{A}_1 e^{-\underline{\gamma} x} - \underline{A}_2 e^{\underline{\gamma} x}\right) \end{aligned}\right\} \qquad 6.63$$

Let's put the origin of the set of coordinate at the receiving end of the line as it's shown in Fig 6,6 When x = 0 y = l and x = l y = 0. Then the new coordinate may be written as y = l-x.

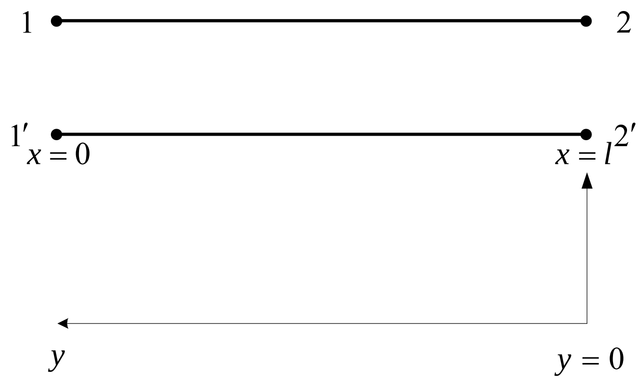



Fig.6.6

From the above set of equations if x=l $V=V_2$ and $I=I_2$

$$\left.\begin{aligned}\dot{V}_2 &= \underline{A}_1 e^{-\underline{\gamma} l} + \underline{A}_2 e^{\underline{\gamma} l}\\ \dot{I}_2 &= \frac{1}{\underline{Z}_C}\left(\underline{A}_1 e^{-\underline{\gamma} l} - \underline{A}_2 e^{\underline{x} l}\right)\end{aligned}\right\} \qquad 6.64$$

from which follows that the integration constants are:

$$\underline{A}_1 = \frac{1}{2}\left(\dot{V}_2 + \underline{Z}_C \dot{I}_2\right)e^{\underline{\gamma} l}; \qquad \underline{A}_{21} = \frac{1}{2}\left(\dot{V}_2 - \underline{Z}_C \dot{I}_2\right)e^{-\underline{\gamma} l}. \qquad 6.65$$

Putting the integration constants of (13.65) in the eq. (13.63) we'll have

$$\left.\begin{aligned}\dot{V} &= \frac{1}{2}\left(\dot{V}_2 + \underline{Z}_C \dot{I}_2\right)e^{\underline{\gamma} y} + \frac{1}{2}\left(\dot{V}_2 - \underline{Z}_C \dot{I}_2\right)e^{-\underline{\gamma} y}\\ \dot{I} &= \frac{1}{\underline{Z}_C}\left\{\frac{1}{2}\left(\dot{V}_2 + \underline{Z}_C \dot{I}_2\right)e^{\underline{\gamma} y} - \frac{1}{2}\left(\dot{V}_2 - \underline{Z}_C \dot{I}_2\right)e^{-\underline{\gamma} y}\right\}.\end{aligned}\right\} \qquad 6.66$$

Using hyperbolic functions yields

$$\left.\begin{aligned}\dot{V} &= \dot{V}_2 ch\underline{\gamma} y + \dot{I}_2 \underline{Z}_C sh\underline{\gamma} y\\ \dot{I} &= \frac{\dot{V}_2}{\underline{Z}_C} sh\underline{\gamma} y + \dot{I}_2 ch\underline{\gamma} y.\end{aligned}\right\} \qquad 6.67$$

In the case of open circuit $I_2=0$ and

$$\left.\begin{aligned}\dot{V}_o &= \dot{V}_2 ch\underline{\gamma} y\\ \dot{I}_o &= \frac{\dot{V}_2}{\underline{Z}_C} sh\underline{\gamma} y.\end{aligned}\right\} \qquad 6.68$$

For all-pass line

$$\dot{V}_o = \dot{V}_2 chj\beta y = \dot{V}_2 \frac{e^{j\beta y} + e^{-j\beta y}}{2} = \dot{V}_2 \cos\beta y$$

$$\dot{I}_o = \frac{\dot{V}_2}{\sqrt{L_0/C_0}} shj\beta y = \sqrt{\frac{C_0}{L_0}}\dot{V}_2 \frac{e^{j\beta y} - e^{-j\beta y}}{2} = \sqrt{\frac{C_0}{L_0}}\dot{V}_2 j\sin\beta y.$$

Corresponding time functions are

$$\left.\begin{aligned}v_o(y,t) &= \sqrt{2}V_2 \cos\beta y \sin\omega t\\ i_o(yt) &= \sqrt{2}\frac{V_2}{Z_C}\sin\beta y \sin(\omega t + 90^o).\end{aligned}\right\} \qquad 65.69$$

The angle of 90° corresponds to the factor "j". the points βy=kπ, where k = 0, 1, 2, …are current nodes and voltage loops.

In the case of short circuited $V_2$ =0 and from the set of equations ( 6.67)

$$\left.\begin{aligned}\dot{V}_s &= \dot{I}_2 \underline{Z}_C sh\underline{\gamma} y\\ \dot{I}_s &= \dot{I}_2 ch\underline{\gamma} y.\end{aligned}\right\} \qquad 6.70$$

For all-pass line

$$\dot{V}_s = \dot{I}_2\sqrt{\frac{L_0}{C_0}}shj\beta y = \dot{I}_2\sqrt{\frac{L_0}{C_0}}\frac{e^{j\beta y}-e^{-j\beta y}}{2} = \dot{I}_2 j\sqrt{\frac{L_0}{C_0}}\sin\beta y$$

$$\dot{I}_s = \dot{I}_2 chj\beta y = \dot{I}_2\cos\beta y$$

Corresponding time functions are

$$\left.\begin{aligned}v_s(y,t) &= \sqrt{2}I_2\sqrt{L_0/C_0}\sin\beta y\sin(\omega t + 90^o)\\ i_s(y,t) &= \sqrt{2}I_2\cos\beta y\sin\omega t.\end{aligned}\right\} \qquad 6.71$$

In this case the points βy=kπ, where k = 0, 1, 2,… are voltage nodes and current loops. Hence the standing voltage wave in a transmission line short circuited at the receiving end is similar to the standing current wave in the same line open circuited.

## Problems:

6.1. The parameters of a coaxial cable are $r_1$=0.25mm, r2=4.5mm. The relative electric permittivity of the insulator used in cable is 5.5. Find the phase velocity and the characteristic impedance of the cable.

6.2. For a transmission line having the length 5km at the frequency 1kHz driving point open circuit and short circuit impedances were defined:

$$\underline{Z}_{in\,o} = 535\angle -64^0\,\Omega;\ \underline{Z}_{in\,sh} = 467.5\angle -10^0\,\Omega.$$

Determine the characteristic impedance and propagation constant.

6.3. A transmission line has the primary parameters:

$$R_0 = 3.75\Omega/km;\ \ L_0 = 4.5mH/km;\ G_0 = 1.5\mu Sm/km;\ C_0 = 8nF/km.$$

Find the value of inductance being switched on after each 1km distance the line to be all pass line.

6.4. The transmission line has the length 10km and the secondary parameters

$$\underline{Z}_c = 500\angle -37^0\,\Omega;\ \underline{\gamma} = 0.1414 + j0.1414km^{-1}.$$

It is short-circuited at the end of the line and gets supply from sinusoidal voltage source at frequency 1kHz. Find voltage and current at starting section of the line if the current at the end of the line $\dot{I}_2 = 1\angle 0^0\,A.$

# 7. Electrical Filters
## 7.1. Types of Filters

Electrical wave filters are two-port networks placed between a source and a load to block or pass a specific range of frequencies. The frequency band transmitted by a filter is termed the pass band of the filter. the frequency band attenuated by a filter is referred to as the stop band of the filter. Extrem frequencies of the band are known as cut of frequencies.

Electrical wave filters are mainly employed in communication systems, in radio engineering, etc. Filters are made up of resistors, inductors and capacitors. At radio frequencies the inductive reactance of an inductor ωL is many times more its resistance. Similarly conductance of a capacitor is many times less then capacitive suseptance G << ωC. So electrical filters, having coils and capacitors may be considered as if consisting of only reactive elements.

Electrical wave filters are usually arranged into symmetrical "T" or "Π" networks of Fig. 7.1. They are two-port networks and for this reason it is possible to use the same concepts of transfer, propagation and constants, phase-shift factor and characteristic impedance that are used for transmission line and two-port network.

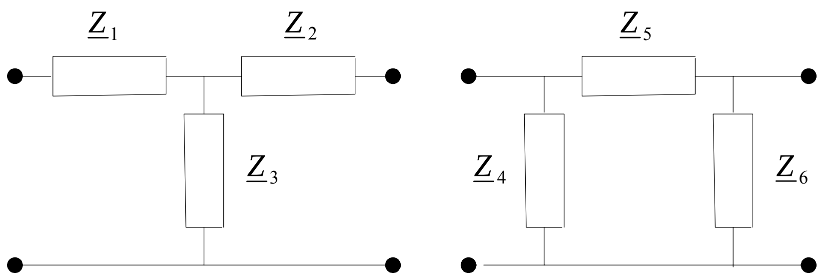


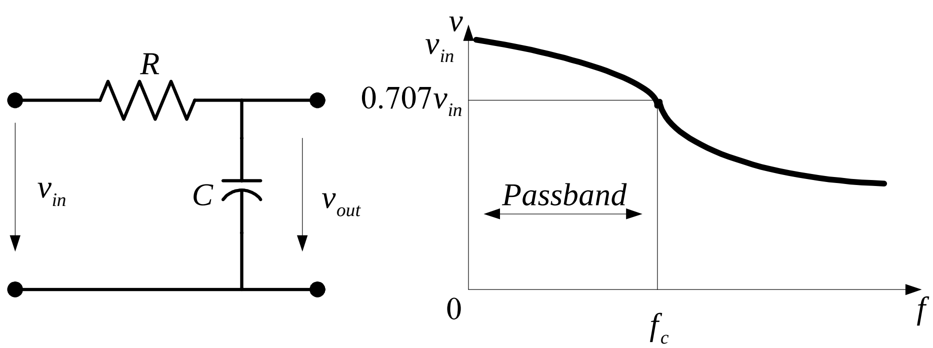

Fig.7.1

Let $Z_1$, $Z_2$ and $Z_3$ in the networks of Fig.7.1 be called series impedances, and $Z_3$, $Z_4$ and $Z_6$ shunt impedances. Then filters for which the product of the series impedance and the respective shunt impedance is constant and independent on frequency are called constant "K" filters. Filters for which this product depends on frequency are called as m-derived filters. In constant "K" filters the load and characteristic impedances are equal, while in m-derived filters they may be unequal.

Depending on the frequency characteristics filters are classified as low-pass, high-pass, band pass and band stop filters. Filters may be designed with fixed or variable parameters. Adjustable filters are used in radio and TV sets to enable the listener or viewer to tune the set to a desired station. Tuning is accomplished by adjusting the resonance frequency of an RLC circuit so that it will pass the desired frequency and reject all others.

## 7.2. Low-pass Filters

Fig.7.2a shows a simple RC circuit used as a low-pass filter, and Fig.7.2b shows the frequency characteristic.




a) b)

Fig.7.2

Frequency $f_c$ is the cutoff frequency; it is the frequency above which the output voltage drops below 0.707 of the input voltage. The bandwidth of the pass band is $BW=f_2-f_1=f_c-0=f_c$.

The filter of Fig.7.2a is a voltage divider for which

$$v_{out} v_{in} \frac{X_c}{\sqrt{R^2 + X_c^2}}. \qquad 7.1$$

At the cutoff frequency $v_{out}$=0.707$v_{in}$ . Substituting it into eq. (7.1) and then solving for $f_c$

$$f_c = \frac{1}{2\pi RC}. \qquad 7.2$$

Example 7.1. Design an RC low-pass filter having a cutoff frequency of 500Hz.
Solution: Assume R=1kΩ. According to eq. 7.2

$$500 = \frac{1}{2\pi C * 1000} \quad \Rightarrow \quad C=318.5nF.$$

## 7.2. High-pass Filters

Fig. 7.3a shows a simple RC circuit used as a high-pass filter, and Fig.7.3b shows the corresponding frequency characteristic. Following the same procedure as used before,

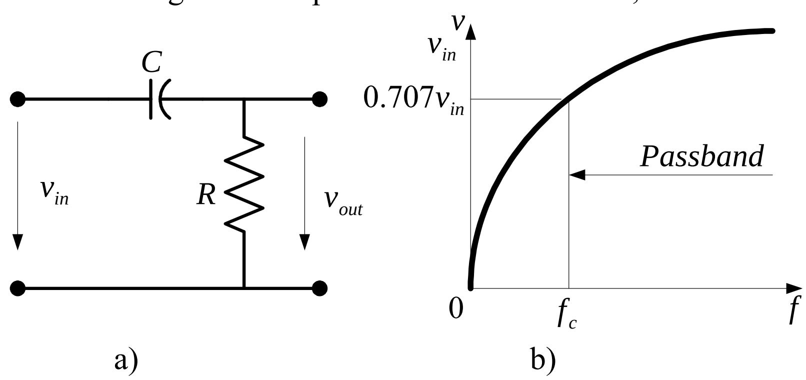



Fig.7.3

$$v_{out} = v_{in}\frac{R}{\sqrt{R^2 + X_c^2}}; \quad 0.707v_{in} = v_{in}\frac{R}{\sqrt{R^2 + X_c^2}} \quad \Rightarrow \quad f_c = \frac{1}{2\pi RC}. \qquad 7.3$$

Therefore, the equation for the cutoff frequency of a high-pass RC filter is identical to that fort the low-pass filter. However, the respective equations for the output voltages are different. The stop band of the filter is $0 < f < f_1$.

## 7.3. Band pass Filters

A band pass filter (circuit diagram and frequency characteristic) that uses a series RLC circuit whose resonance frequency and bandwidth provide the desired pass band is shown in Fig.7.4.

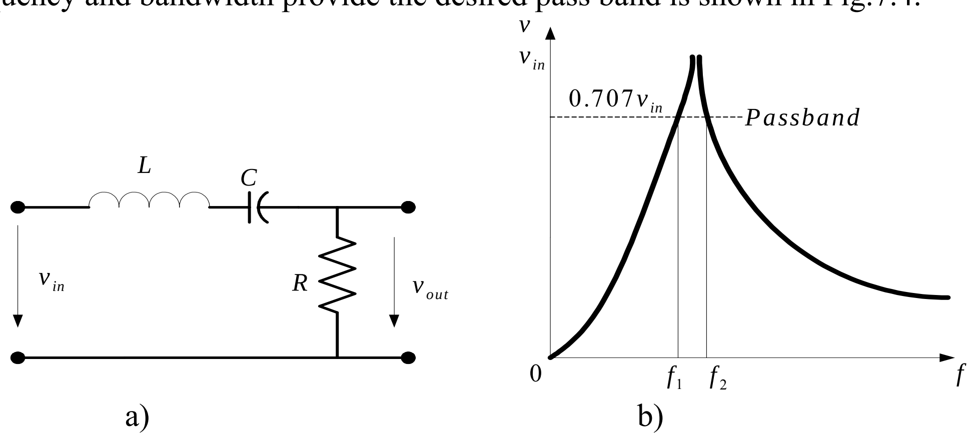



Fig.7.4

The main relationships of the circuit of Fig.7.4a are:

$$f_c = \frac{1}{2\pi\sqrt{LC}}, \qquad Q_s = \frac{2\pi f_r L}{R}, \quad v_{out} = v_{in}\frac{R}{\sqrt{R^2 + (X_L - X_c)^2}} \qquad 7.4$$

A typical frequency – response curve for a band pass filter is shown in Fig.7.4b.The cutoff frequencies are $f_1$ and $f_2$ and band stops are $0 < f < f_1$ and $f_2 < f < \infty$.

Example 7.2. Assuming the circuit parameters in Fig. 7.4a are L=50mH, C=127nF, R=57Ω, determine (a) the resonance frequency; (b) the bandwidth; (c) the cutoff frequencies.

Solution:

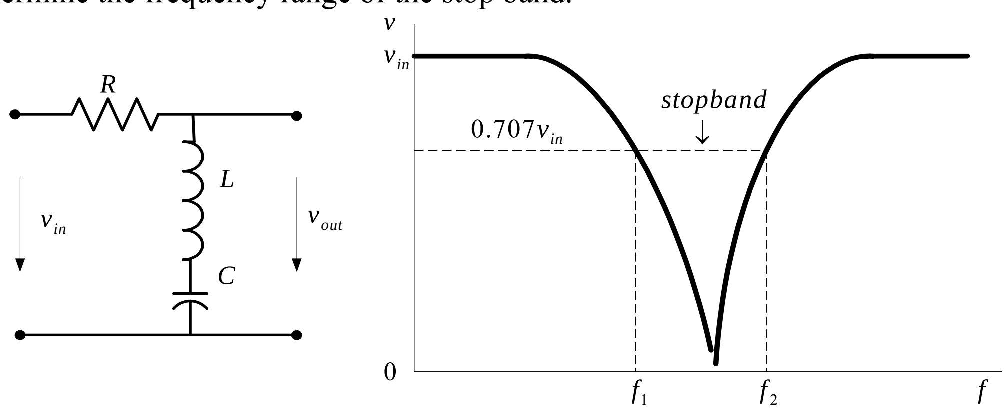
(a) $f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{50*10^{-3}*127*10^{-9}}} = 2kHz.$

(b) $Q_s = \frac{\omega_r L}{R} = \frac{2\pi*50*10^{-3}}{57} = 11. \qquad BW = \frac{f_r}{Q_s} = \frac{2000}{11} = 182Hz.$

(c) $f_1 = f_r - \frac{BW}{2} = 2000 - \frac{182}{2} 1909Hz; \quad f_2 = f_r + \frac{BW}{2} = 2000 + \frac{182}{2} = 2091Hz.$

## 7.4. Band stop Filters

The band stop filter shown in Fig. 7.5 uses a series RLC circuit whose bandwidth and resonance frequency determine the frequency range of the stop band.




a) b)

Fig. 7.5

A typical frequency-response curve for a band stop filter is shown in Fig.7.5b. The stop bands are $0 < f < f_1$ and $f_2 < f < \infty$ .For input frequencies in the stop band region $v_{out} < 0.707v_{in}$. The resonance frequency and bandwidth of the filter may be determined from

$$f_r = \frac{1}{2\pi\sqrt{LC}}, \qquad BW = \frac{f_r}{Q_s} = \frac{R}{2\pi L}. \qquad 7.5$$

Example 7.3. Determine (a) the capacitance required for a band stop filter of Fig.7.5a that will block 85kHz if R=2000Ω and L=60mH.(b) Determine the bandwidth.

Solution:

a) $f_r = \frac{1}{2\pi\sqrt{LC}}. \quad \Rightarrow C = \frac{1}{f_r^2 4\pi^2 L} = \frac{1}{(85000)^2 4\pi^2 60*10^{-3}} = 58pF.$

b) $Q_s = \frac{2\pi f_r L}{R} = \frac{2\pi 85000*60*10^{-3}}{2000} = 16. \quad BW = \frac{f_r}{Q_s} = \frac{85000}{16} = 5.3kHz.$

**Problems:**

7.1. Design an RC low-pass filter that will have a cutoff frequency 800Hz. Assume R=2kΩ.
7.2. Design an RC low-pass filter that will have a cutoff frequency 2000Hz. Assume C=80nF.
7.3. Design a high-pass RC filter that uses a 650pF capacitor and has a cutoff frequency of 8kHz.
7.4. Design a series-resonance-type band pass filter that has cutoff frequencies of 15kHz and 25kHz. The coil has an inductance of 50mH and negligible resistance.
7.5. Determine the required capacitance for a series-resonance band stop filter that will be lock 50kHz. Assume R=1.5kΩ and L=55mH.